BULLETIN

OF THE

AMERICAN MATHEMATICAL SOCIETY

A HISTORICAL AND CRITICAL REVIEW
OF MATHEMATICAL SCIENCE

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NOTICE TO AUTHORS

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1. That all manuscripts be typewritten.

2. That authors notice and employ the styles used in selection of type, in arrangement, in headings, in references, etc.

3. That a carbon copy with all formulas and corrections be kept by the author. A second carbon copy, to be kept by the editors, would be appreciated.

4. That greater care than formerly be exercised in spelling, punctuation, handwritten symbols, etc.

Manuscripts that do not fulfill these requirements, which are intended primarily as a safeguard, may be returned to the authors for revision.

THE EDITORS

Dec. 1, 1923

THE THIRTIETH SUMMER MEETING OF THE AMERICAN MATHEMATICAL SOCIETY

The thirtieth summer meeting of the Society was held at Vassar College, Poughkeepsie, New York, on Thursday and Friday, September 6–7, 1923, the college opening its buildings for the entertainment of the visitors and thus contributing greatly to the success of the sessions. The Mathematical Association of America held its meetings immediately preceding and a joint session of the two societies was arranged for Thursday afternoon, at which addresses were delivered by Professor L. J. Mordell, of the University of Manchester, and Professor R. C. Archibald, of Brown University. At the joint dinner on Thursday evening, President McCracken of Vassar College welcomed the visiting mathematicians and spoke on the relation of the undergraduate college to research. Tea was served every afternoon after the sessions at the home of Professor and Mrs. H. S. White. At the last session, it was voted to express to Vassar College and to the department of mathematics the thanks of the Society for their generous hospitality and for the admirable arrangements.

The attendance included the following seventy-nine members of the Society:

Agard, Archibald, G. N. Armstrong, Babb, Bacon, Barney, E. R. Beckwith, S. R. Benedict, Bernstein, E. W. Brown, H. S. Brown, J. A. Bullard, W. G. Bullard, R. W. Burgess, Cairns, G. M. Conwell, Copeland, Court, L. D. Cummings, Dimick, Dines, Eiesland, Feldstein, W. B. Ford, Glenn, M. C. Graustein, W. C. Graustein, Grove, E. R. Hedrick, A. M. Howe, Huntington, Ingels, Kazarinoff, Kline, W. D. Lambert, F. P. Lewis, Luck, McCain, McDonnell, MacDuffee, Mathews, H. H. Mitchell, C. N. Moore, Mordell, Morenus, C. C. Morris, Nassau, Pierpont, Pitcher, Post, Rambo, Ranum, Reddick, L. J. Reed, R. G. D. Richardson, E. D. Roe, J. R. Roe, Rosenbach, Seely, Sinclair, Slaught, Slobin, C. E. Smith, D. E. Smith, Gertrude Smith, W. M. Smith, M. J. Sperry, B. M. Turner, Tyler, Vandiver, Veblen, M. E. Wells, H. S. White, Whittemore, Constance Wiener, Norbert Wiener, E. W. Wilson, Worthington, J. W. Young.

At the meeting of the Council, the following twenty-seven persons were elected to membership in the Society:

Professor Samuel Latimer Boothroyd, Cornell University;
Mr. Leonidas Hamlin Bunyan, University of Wisconsin;
Miss May Bryan Carter, Western College;
Professor Jacques Chapelon, University of Lille;
Rev. Gabriel Hippolytus Collignon, Loyola University;
Mr. Paul Sumner Dwyer, Pennsylvania State College;
Miss Lois Wilfred Griffiths, University of Washington;
Professor Lilian May Hackney, Marshall College;
Professor George Wellman Hess, Whitworth College;
Mr. Arthur Owen Hickson, Brown University;
Captain Grafton S. Kennedy, Ordnance Department, United States Army;
Mr. Harry Raymond Kimball, San Francisco, Calif.;
Mr. Harry Levy, Princeton University;
Dr. Elsie Jeannette McFarland, University of California;
Mr. Robert Houghton Marquis, Pennsylvania State College;
Professor Rhesa Lancaster Newlin, Guilford College;
Professor Niels Erik Nörlund, University of Copenhagen;
Mr. Howard Percy Robertson, University of Washington;
Mr. Henry Allen Robinson, Johns Hopkins University;
Professor Lao Genevra Simons, Hunter College;
Mr. Marshall Harvey Stone, Harvard University;
Mr. John Clement Tinner, Chicago, Ill.;
Lieutenant Lawrence Wainwright, United States Navy;
Professor Benjamin Lewis Waits, Alcorn College;
Professor Anne Marie Whelan, Olivet College;
Miss Evelyn Prescott Wiggin, Brown University;
Mr. Hugh Herbert Wolfenden, Grimsby, Ontario.

Forty-nine applications for membership were received. The Secretary announced that the following members of the London Mathematical Society had accepted membership under the reciprocity agreement since the April meeting.

Mr. Charles G. F. James, London;
Professor Louis Joel Mordell, University of Manchester.

The Council announced the appointment of the following committees: Professors Tyler, E. R. Hedrick, R. G. D. Richardson, Slaught, Veblen, H. S. White and J. W. Young on revision of the by-laws; Professors Eisenhart, Birkhoff and Dickson on nomination of officers; Professors Veblen, Birkhoff, Eisenhart, E. R. Hedrick, Slaught and J. W. Young on printing of the journals. Professor Birkhoff represented the Society at the inauguration of President Stratton of the Massachusetts Institute of Technology on June 11, 1923.

It was announced that the Guarantee Fund for the expenses

of the endowment campaign had been raised and that the campaign itself would be formally opened in October. The whole-hearted cooperation of the members of the Society was invited. The committee on incorporation reported that it was confidently expected that the necessary legal forms could be attended to so that the incorporated body would supersede the unincorporated before the end of the year.

A resolution was adopted sanctioning the establishment of a lectureship to be known as the Josiah Willard Gibbs Lectureship, the lecture to deal in semi-popular form with some aspect of mathematics or its applications. The President was authorized to appoint a committee to make arrangements for the first lecture, which will probably be given in New York City during the winter of 1923–24.

In view of the anticipated meeting of the International Mathematical Congress in Canada in the summer of 1924 the Society decided to omit its summer meeting for that year.

At the meeting of the Society and the Association it was voted to request the Secretaries of the two organizations to send a letter to the Physico-Mathematical Society of Japan, expressing the sympathy of American mathematicians for their colleagues in Japan under the calamity that has befallen their country through the great earthquake.

At the joint session the following papers were read:

I. *An introductory account of the arithmetical theory of algebraic numbers and its recent development*, by Professor L. J. Mordell. (Address delivered at the request of the American Mathematical Society.)

II. *Mathematicians and music*, by Professor R. C. Archibald. (Address of the retiring President of the Mathematical Association of America.)

President Oswald Veblen presided at the regular sessions of the Society, relieved by Professor H. S. White. Titles and abstracts of the papers read at these sessions follow below. The papers of Professor R. L. Moore, Dr. Murray, Professor Altshiller-Court, Professor Murnaghan, Dr. Walsh, Mr. Thomas, Mr. McDonnell, Mr. Michal, and Professor Evans

were read by title; Mr. Rainich was introduced by Professor Kasner, and Mr. Michal by Professor Evans.

1. Professor H. S. White: *Note on five points and a cyclic correspondence.*

Five given numbers in a given cyclic order determine completely a (2, 2) correspondence. The explicit equation is deduced, and an algebraic proof of its periodicity.

2. Professor B. A. Bernstein: *A generalization of the syllogism.*

This paper will appear in full in an early issue of this BULLETIN.

3. Professor B. A. Bernstein: *Operations with respect to which the elements of a boolean algebra form a group.*

Let a' denote the negative of a boolean element a, $a + b$ the logical sum of a and b, and ab their logical product; the author pointed out in a previous paper that $ab' + a'b$ and $ab + a'b'$ are operations with respect to each of which the elements of a boolean algebra form an abelian group. In this paper he determines all the operations of a boolean algebra with respect to which the elements form a group in general and an abelian group in particular.

4. Mr. H. S. Vandiver: *A new type of criteria for the first case of Fermat's last theorem.*

By transforming the criteria of Kummer, the author shows that if $x^p + y^p + z^p = 0$ is satisfied in integers x, y, and z, not zero and prime to the odd prime p, then

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{u^2} \equiv 0 \pmod{p},$$

where u is the greatest integer in $p/3$. The paper will be offered for publication to the ANNALS OF MATHEMATICS.

5. Mr. H. S. Vandiver: *A method for finding a factor of an integer of the form* $8n + 1$.

If an integer m of the form $8n + 1$ is the product of two prime factors, then each factor has the same residue modulo 8. The author shows that m may be represented in at least two different ways in one of the forms $x^2 + y^2$, $x^2 + 2y^2$, $2x^2 - y^2$, where $x < \sqrt{p}$, $y < \sqrt{p}$. The factors of m are found from the different representations. The paper will be offered for publication to this BULLETIN.

6. Mr. G. Y. Rainich: *The distribution of primes and the finiteness of the number of discriminants with a given number of classes.*

In this paper a proof of the following proposition is given: the number of integers m for which the ratio of the whole number of primes between m and m^2 to the number of primes of the form $x^2 + xy + my^2$ between the same limits does not exceed a given value is finite. It seems probable that when the number of classes of the discriminant $1-4m$ is fixed the aforesaid ratio does not exceed a certain value (which is very near to the double of the number of classes); should this be proved, the finiteness of the number of discriminants with a given number of classes would be established. The degree of non-unicity of decomposition in prime factors is characterized by the maximum number of factors in a product which is divisible by a prime without any partial product being divisible by a prime; the connection of this constant in the case of quadratic fields with the structure of the group of classes is shown. It is proved that when the number of classes is unity all the numbers $x^2 + xy + my^2$ below m^2 are primes (for x and y relative primes).

7. Professor O. E. Glenn: *A complete system of differential parameters of orders < 3 of the binary differential cubic.*

This paper, combined with two communicated previously as preliminary reports (this BULLETIN, April and May, 1923), forms a memoir entitled *On the reduction of differential parameters in terms of finite sets, with remarks concerning differential invariants of analytic transformations*, which has been offered to the AMERICAN JOURNAL OF MATHEMATICS. The finite system determined for the cubic belongs to the domain of rational polynomials in derivatives of orders $\leqq 3$ of the functions $x_i = \varphi_i(y_1, y_2)$ $(i = 1, 2)$ of the transformations.

8. Professors E. R. Hedrick and Louis Ingold: *Analytic and non-analytic functions in three dimensions.*

In a paper by Hedrick, Ingold, and Westfall, soon to appear in the JOURNAL DE MATHÉMATIQUES, it is shown that analytic functions of a complex variable may be distinguished from non-analytic functions by the fact that the Tissot indicatrix becomes a circle in the analytic case; also by the fact that certain characteristic directions become indeterminate. In the present paper the authors show that for transformations of space into itself, there is an ellipsoid at each point which plays a rôle

analogous to that of the Tissot indicatrix in the plane, and there are at each point characteristic directions. In particular they study those transformations for which this ellipsoid reduces to a sphere, and the characteristic directions become indeterminate. It is found that these transformations are analogous to analytic functions. The transformations coincide with conformal transformations of space, and the set of functions defining them satisfy equations analogous to the Cauchy-Riemann equations; also each of these functions satisfies an equation analogous to Laplace's equation.

9. Professor R. L. Moore: *A connected and regular point set which contains no arc.*

It is shown that there exists a connected point set which is regular (connected im kleinen) at every one of its points but which contains no simple continuous arc.

10. Professor R. L. Moore: *Concerning the sum of a countable infinity of continua in the plane.*

Sierpinski has shown that there exists no bounded plane continuum which is the sum of a countable infinity of mutually exclusive closed point sets. He has raised the question whether this result would still hold good if the word "bounded" were omitted. In the present paper it is shown that, for the special case where the closed point sets in question are all continua, this question may be answered in the affirmative.

11. Professor R. L. Moore: *A continuum considered as the sum of its prime elements.*

Hans Hahn has recently introduced the notion of the prime elements (Primteile) of a continuum and has proved a theorem which may be interpreted to mean that, with respect to its prime elements considered as points, every continuum which is irreducible between two points is a simple continuous arc which has those points as end points. In the present paper it is shown, among other things, that every bounded continuum whatsoever is a continuous curve with respect to its prime elements considered as points.

12. Professor Mary E. Sinclair: *The brachistochrone with variable end points.*

Conditions for fixed end points require that the extremals be single arches of inverted cycloids. The first derivatives of the extremal-integral, $J(u, v)$, where u and v are the parameters of two fixed plane curves on which the end points vary,

gives the known transversality conditions. Its second derivatives give conditions involving the curvatures of the fixed curves and the parameters of the cycloid at the end points, such that if but one end point vary, its focal point on the arch may be studied. If both end points vary, the second derivatives give inequalities which may be interpreted as restricting the curvature of one of the fixed curves when that of the other and the parameters of the cycloid are given. A geometric interpretation of these results is given. Sufficient conditions for a relative minimum follow from the theorem of Hahn used by Merrill in his generalization of the problem of Dido.

13. Professor R. G. D. Richardson: *A new necessary condition for relative extrema in quadratic and hermitian forms.*

If $A = \sum_{ij}^{1,n} a_{ij}x_jx_i$, $B = \sum_{ij}^{1,n} b_{ij}x_jx_i$ be real quadratic forms of which B is positive definite, the minimum of A subject to the condition $B = 1$ must be one of the n solutions $x_1^{(i)}, \cdots, x_n^{(i)}$ of the equations $\sum_j (a_{ij} - \lambda b_{ij})\, x_j = 0$, corresponding to one of the n zeros λ_i of the determinant $|a_{ij} - \lambda b_{ij}|$. The new necessary condition for a minimum obtained by the author determines the particular solution furnishing the minimum. For the λ_i which is desired, it is necessary that in the determinant $|a_{ij} - \lambda_i b_{ij}|$ the sequence σ_i $(i = 0, \cdots, n)$ of sums of all the principal minors obtained by deleting i rows and the same i columns show no permanence of sign. This condition together with those already known forms a sufficient set.

More generally, when the linear conditions $\sum_{ij} b_{ij}\, x_j^{(1)} x_i = 0, \cdots, \sum_{ij} b_{ij} x_j^{(m)} x_i = 0$ are added to the problem, the minimum is again a solution of the same linear equations and a similar new necessary condition on minors of the bordered determinant completes the set of sufficient conditions.

The same theory applies also in the case of hermitian forms.

14. Professor C. N. Moore: *On the summation of trigonometric series by Euler's method.*

In this paper it is shown that while Euler's method of summation will serve to sum certain divergent trigonometric series, it is less effective than Cesàro's method when applied to Fourier's series. For it follows from a general criterion due to Lebesgue that Euler's method will not serve to sum at points of continuity of the function developed all the Fourier's series corresponding to functions having a Lebesgue integral, whereas it is a well known theorem that Cesàro's method has this property.

15. Professor L. L. Dines: *Concerning a suggested and discarded generalization of the Weierstrass factorization theorem.*

This paper will appear in full in an early issue of this BULLETIN.

16. Professor L. L. Dines: *A theorem on the factorization of polynomials of a certain type.*

Polynomials of the form $P \equiv y^n + P_{n-1}(x)y^{n-1} + \cdots + P_1(x)y + P_0(x)$, where the coefficients $P_j(x)$ are power series converging in a sufficiently restricted neighborhood of the origin and vanishing with x, are of importance in the theory of implicit functions. In the present paper, the author gives a direct algebraic proof of the following algebraic theorem relative to such polynomials: *If the characteristic line (Newton polygon) of the polynomial P is a broken line of k segments, then P is a product of k polynomials of the same type as P, each of which has an unbroken characteristic line the same in length and direction as one segment of the characteristic line of P.* The theorem is for this type of polynomials a converse of the "Theorem on the Product" proved by Blumberg (TRANSACTIONS OF THIS SOCIETY, vol. 17, p. 530) for a very general class of polynomials defined postulationally.

17. Mr. Donat Kazarinoff: *The scientific work of A. M. Liapounoff.*

This paper gives an account of the scientific work of the eminent Russian mathematician A. M. Liapounoff (born in Yaroslavl, 1857; died in Odessa, 1918). His achievements won wide recognition; he was a member of the Russian Academy of Sciences, a foreign member of the Accademia dei Lincei, a correspondent of the Académie des Sciences, etc. His basic contributions are on the figures of equilibrium of a homogeneous rotating liquid.

The present paper depends for its information almost entirely upon the obituary memoir by V. A. Stekloff, BULLETIN OF THE RUSSIAN ACADEMY OF SCIENCES, ser. 6, vol. 13 (1919), pp. 367–388.

18. Dr. Norbert Wiener: *The quadratic variation of a function.*

In this paper, a discussion of the second degree analog of the total variation of a function is employed to establish certain inequalities connecting the total variation of a function, its saltus, and its Fourier coefficients.

19. Dr. F. H. Murray: *Certain orbits with arbitrary masses in the problem of three bodies.*

This paper is devoted chiefly to the study of certain special cases of the problem of three bodies, by methods in which the notion of "system of invariant relations" as defined by Poincaré plays an important rôle. After certain properties of differential systems which admit a set of invariant relations have been developed, some qualitative results are obtained concerning isosceles triangle solutions with axis of symmetry. The equations of the problem of three bodies in the plane are given in a comparatively simple form, with the aid of which solutions asymptotic to the straight line and equilateral triangle solutions, respectively, are obtained. These results complete, on certain points, the researches of D. Buchanan concerning these solutions.

20. Professor W. C. Graustein: *Applicability with preservation of both curvatures.*

This paper will appear in full in an early issue of this BULLETIN.

21. Professor W. C. Graustein: *Isometric W-surfaces.*

This paper contains a complete classification of isometric W-surfaces and a discussion of the new types found. Special attention is given to isometric W-surfaces admitting continuous deformations into themselves.

22. Mr. G. Y. Rainich: *A new kind of representation of curved space.*

Given a surface S and a plane P we draw through a point A of S the normal to S and the perpendicular to P. We call the intersection of the bisector of the angle formed by these two lines with P the generalized stereographic projection of A. Vectors belonging to S (cf. an article by the author in the PROCEEDINGS OF THE NATIONAL ACADEMY, June, 1923) can also be projected stereographically, and from this arises a new kind of representation of tensors belonging to the surface through tensors of a plane, which has an advantage over the usual representation so far as algebraic relations are concerned, since no such distinctions as that between covariant and contravariant quantities are introduced. Differentiation, on the other hand, is more complicated in this system. Projecting the neighborhood of a point stereographically on the tangent plane at this point, and using cartesian coordinates in this

plane, we obtain "semi-cartesian" coordinates for the representation of the surface. We can use a similar representation for higher spaces and also for curves on a plane.

23. Professor Nathan Altshiller-Court: *On two circles.*

The pair of lines joining the two points of intersection of two circles to a variable point on one of these circles determine in the second circle a chord of constant length. This length x is given by the proportion $x : s = d : p$, where s, d, p are the lengths of the common chord, the line of centers, and the radius of the first circle respectively. The author applies this result to a coaxial system of circles, to various circles connected with the triangle, etc., and gives a projective generalization of the proposition.

24. Professor F. D. Murnaghan: *A minimum problem in elementary geometry.*

The problem of finding the point the sum of whose distances to four points in space is a minimum was proposed by Steiner and has been considered by many geometers, notably by R. Sturm. Although Steiner gave a hint of the proper mode of attack, his hint seems to have escaped attention. It is apparent that if planes each through one of the four points perpendicular to its join to a fifth point form an equifacial tetrahedron, this fifth point will make $r_1 \pm r_2 \pm r_3 \pm r_4$ a minimum or a maximum. To find such a fifth point x (using as coordinates perpendiculars on the faces of the tetrahedron formed by the four given points) note that the pedal tetrahedron of its isogonal conjugate $y = 1/x$ will also be equifacial, and hence have its opposite edges equal. There are therefore 8 points y, the base points of the net of quadrics $y_1^2 + y_2^2 + 2c_{12}y_1y_2 = y_3^2 + y_4^2 + 2c_{34}y_3y_4$, $y_1^2 + y_3^2 + 2c_{13}y_1y_3 = y_2^2 + y_4^2 + 2c_{24}y_2y_4$, $y_1^2 + y_4^2 + 2c_{14}y_1y_4 = y_2^2 + y_3^2 + 2c_{23}y_2y_3$, where c_{rs} is the cosine of the internal angle between the faces r and s of the tetrahedron of reference.

25. Dr. J. L. Walsh: *A generalization of evolutes.*

Let points P_1 and P_2 trace two curves C_1 and C_2 so that the tangents to C_1 and C_2 at P_1 and P_2 are parallel. Let the point P be determined so that it divides the segment P_1P_2 in the constant ratio $m_1 : m_2$; P traces a curve C. Then the centers of curvature Q_1, Q_2, Q of C_1, C_2, C at corresponding points P_1, P_2, P are such that Q divides the segment Q_1Q_2 in the ratio $m_1 : m_2$. The circles of curvature of C_1, C_2, C at P_1, P_2, P have as common center of similitude the point of tangency of the variable

line P_1P_2 with its envelope. Consider the family of curves found by varying the ratio $m_1 : m_2$. An infinity of curves of this family can be traced by points of an elastic string which moves so as to wrap itself on the envelope of the line P_1P_2.

26. Mr. J. M. Thomas: *Congruences of circles studied with reference to the surface of centers.*

The surface of centers S, rather than the envelope of the planes, is used as the framework of reference. Certain theorems relating to S in general are first proved. The main part of the paper is devoted to congruences C characterized as follows: each point P of a circle of the congruence is on a surface whose tangent plane at P contains the radius to P and cuts the plane of the circle at an angle $90° - \varphi$, where φ is a function at most of the two parameters of the circle. Moreover, for a congruence C the surface S is distinct from the envelope of the planes. When all the circles have the same radius and φ is constant, it is found that the planes of the circles are the osculating planes of a single parameter family of curves of constant torsion on S. The case where S degenerates into a curve is considered. For a cyclic system the curve of centers must be plane unless the circles are in the normal planes to the curve.

27. Professor John Eiesland: *A theorem in relativity.*

The following theorem is proved: Necessary and sufficient conditions that a centro-symmetric space with line element $-ds^2 = \varphi_2 dr^2 + \varphi_3(d\theta^2 + \sin^2\theta d\varphi^2) - \varphi_1 dt^2$, φ_1, φ_2 and φ_3 being arbitrary functions of r and t, shall be reducible to the static form are $\varphi_1\varphi_2 G_1^0 = (\partial\varphi_3/dr)(\partial\varphi_3/dt)$, and

$$\varphi_1\varphi_2\varphi_3(G_1 - G_0) = -\psi'\left[\varphi_1\left(\frac{d\varphi_3}{dr}\right)^2 + \varphi_3\left(\frac{d\varphi_3}{dt}\right)\right],$$

where ψ is an arbitrary function of φ_3. It follows that the Einstein solar field is necessarily static,* and that the same is true of other spaces of importance in relativity theory.

28. Professor John Eiesland: *On a generalization of Kummer's surface in odd n-space.*

Adopting as space element the ∞^n flats $\sigma_0 x_i - \sigma_{n/2} y_i + \rho_i = 0$, $\sigma_0 z + \sum\sigma_i y_i + \rho_{n/2} = 0$, $\sigma_{n/2} z + \sum\sigma_i x_i - \rho_0 = 0$ [in

* For an analytic proof of this proposition based on the existence-theorem for a pair of differential equations, see Birkhoff, *Relativity and Modern Physics*, Cambridge, 1923, pp. 253–256.

S_{n-1}, where $\sum \rho_i \sigma_i = 0$, the author considers the quadratic flat-complex $\varphi = \sum a_i(\rho_i^2 + \sigma_i^2) + 2\sum b_i \rho_i \sigma_i = 0$, and the singular surface of this complex. This surface is of the eighth order and eighth class, it is self dual, and has a 2-spread for locus of double-spreads. If Klein's coordinates $y_{2i+1} = \rho_i + \sigma_i$, $iy_{2n+2} = \rho_i - \sigma_i$ are introduced, the complex takes the form $\sum_1^{n+1} y_i^2 = 0, \sum_1^{n+1} k_i y_i^2 = 0$. The coordinates of the flats y_i may be represented as hyperelliptic functions of n parameters λ_i, and therefore also the coordinates of the singular surface may be similarly represented.

29. Mr. John McDonnell: *A note on chapter* 2 *of volume* 3 *of L. E. Dickson's History of the Theory of Numbers.*

In the above-mentioned chapter various theorems are stated concerning the expressibility of $4p$ in the form $a^2 + 27b^2$ when p is a prime $= 6n + 1$. In the present paper it is proved that p itself is expressible in the form $a^2 + 27b^2$ if $2^{(p-1)/3} \equiv 1 \pmod{p}$ and conversely.

30. Mr. A. D. Michal: *Integro-differential invariants of one-parameter groups of Volterra transformations.*

The author continues the discussion of integro-differential invariants, in particular functionals of the form $f[\underset{0}{\overset{1}{y}}, \underset{0}{\overset{1}{y'}}]$ which are "analytic," and shows that a necessary and sufficient condition that such a functional be invariant of a one-parameter group of linear functional transformations of Volterra type is that it satisfy a certain completely integrable equation in partial functional derivatives. The expansion of the solution of this equation is obtained.

31. Professor G. C. Evans: *The dynamics of monopoly.*

The author considers a problem in the dynamics of economics, namely, the change from one price to another under conditions of monopoly, when the demand function involves the rate of change of price as well as the price itself. The total profit over the stretch of time involved in the change shall be a maximum. The demand function is taken as of the first degree in the price and the rate of change of the price, and the cost is assumed to be a quadratic function of the amount produced.

R. G. D. RICHARDSON,
Secretary.

AN INTRODUCTORY ACCOUNT OF THE ARITHMETICAL THEORY OF ALGEBRAIC NUMBERS AND ITS RECENT DEVELOPMENTS*

BY L. J. MORDELL

1. *Introduction.* In dealing with the subject of my lecture, I might have considered it from a purely logical point of view, and developed it in all its beauty in this manner. I think, however, it will be of more interest to you if I introduce it from the historical standpoint. Its beginnings date from Euler, who attempted to prove Fermat's statement, that the only integer solutions of the equation $y^2 + 2 = x^3$ were $x = 3$, $y = \pm 5$, by putting $x = a^2 + 2b^2$ and taking

$$y + \sqrt{-2} = (a + b\sqrt{-2})^3.$$

By equating irrational parts, he found

$$1 = b(3a^2 - 2b^2),$$

whence $b = 1$, $a = \pm 1$; but it is neither obvious nor true in general that all the integer solutions can be found in this way, —one used by Euler and Lagrange for some related questions. Then, about 1800, much interest was shown in the so-called law of quadratic reciprocity, first rigorously proved by Gauss; namely, that if p and q are two odd positive primes

$$\left(\frac{p}{q}\right)\left(\frac{q}{p}\right) = (-1)^{\frac{p-1}{2}\cdot\frac{q-1}{2}}.$$

The symbol (p/q) denotes $+1$ or -1, according as the congruence $x^2 \equiv p \pmod{q}$ is possible or impossible, and then p is called a quadratic or non-quadratic residue respectively of q. With certain extensions, this law is really equivalent to a reduction formula enabling us to calculate the value of the symbol (p/q), and forms the foundation of the theory of numbers.

* Lecture read before the London Mathematical Society on January 18, 1923, and, by request of the program committee, before the American Mathematical Society, on September 6, 1923.

Obvious generalizations are suggested for other congruences, such as

$$x^3 \equiv p \pmod{q}, \qquad x^4 \equiv p \pmod{q}.$$

Taking the latter, it was found that the result appeared in a very complicated form, which, however, simplified remarkably if, instead of primes p and q, we considered their decompositions (when possible) in the form

$$p = a^2 + b^2,$$

or into complex factors $(a + ib)(a - ib)$. This led Gauss to the study of the arithmetic properties of complex numbers of the form $z = a + ib$, where a and b are integers. He proved that they did not differ essentially from those of ordinary integers, and showed that it was a very simple matter to define primes so that complex numbers could be factored uniquely, that if a product z_1z_2 were divisible by a complex prime z, then either z_1 or z_2 was divisible by z, etc.

Gauss also showed that if p was a prime of the form $4n + 1$, then the value of a (odd say) was given by the absolutely least residue satisfying the congruence

$$a \equiv (-1)^{n+1}\tfrac{1}{2}\frac{2n!}{(n!)^2} \pmod{p}.$$

This result was extended by Stern, Cauchy, Jacobi, and Eisenstein. The proofs depended upon complex numbers formed from other roots of unity than the fourth root i and applications were made to laws of reciprocity.

The most important facts concerning these numbers were discovered in connection with Fermat's last theorem on the insolubility in non-zero integers of the equation

$$x^p + y^p = z^p.$$

The left-hand side can be factored in the form

$$(x + y)(x + \zeta y)(x + \zeta^2 y) \cdots (x + \zeta^{p-1} y) = z^p,$$

where ζ is a complex pth root of unity; and it appeared to be a natural assumption, by analogy with elementary arithmetic, to put

$$x + \zeta y = (A + B\zeta + C\zeta^2 + \cdots)^p,$$

or perhaps to some multiple of the right-hand side, where $A, B, C, \cdots$ are ordinary integers. This involved the assumption that these algebraic numbers could be factored in a unique manner, an assumption which was, however, in general erroneous. This can be seen from a far more simple case:

$$21 = 3 \cdot 7$$
$$= (4 + \sqrt{-5})(4 - \sqrt{-5}) = (1 + 2\sqrt{-5})(1 - 2\sqrt{-5}),$$

and it is easily verified that none of $3, 7, 4 \pm \sqrt{-5}, 1 \pm 2\sqrt{-5}$ can be split up into factors of the form $a + b\sqrt{-5}$ with a and b integers. Again

$$3^2 = (2 + \sqrt{-5})(2 - \sqrt{-5})$$

and $2 \pm \sqrt{-5}$ are not squares, and neither have a common factor of the form $a + b\sqrt{-5}$, nor can be split into factors of this form.

The primary object of the theory was to re-establish order in the chaos produced by the breakdown of the fundamental theorem upon which depends all the higher arithmetic. This was accomplished by Kummer in the special case of the algebraic numbers arising from the roots of unity, and more generally by several other writers. We shall give an account of Dedekind's method, since the main idea is not only easily explained, but is also very characteristic of mathematics, in generalizing a concept or a function by including it in a wider one. This idea is familar to all, e.g., $n!$ initially defined by $1 \cdot 2 \cdot 3 \cdots n$ when n is an integer, is generalized to

$$\Gamma_{(n)} = \int_0^\infty e^{-x} x^{n-1} dx,$$

when n has its real part positive; and for all values of n by the well known infinite product. Again the chord of contact of the tangents from a point P to a conic can be generalized as the polar of P, giving from one point of view a simpler interpretation when the tangents are imaginary. Dedekind's idea was to consider groups of numbers which he called ideals, and with the obvious method of multiplying and dividing such groups as suggested by their definition, he showed that a

unique factor law existed for his ideals, and that his results included as particular cases the fundamental laws of arithmetic and the correct arithmetical deductions to be drawn from relations involving algebraic numbers.

2. *Algebraic Numbers.* We must now define an algebraic number. The number θ is called an *algebraic number* if it is the root of an equation of the form

$$a\theta^n + b\theta^{n-1} + \cdots + l = 0,$$

where $a, b, \cdots, l$ are ordinary integers. If $a = 1$, θ is called an *algebraic integer*, and it is easy to show that this generalization is consistent, e.g., the sum, difference, product of integers are integers, and that if an algebraic integer is rational, it must be an ordinary integer. We may suppose that the equation in θ is irreducible in the field of rationality R defined by the ordinary integers, and that it has r_1 real roots and r_2 pairs of imaginary roots.

Any rational function of θ with rational coefficients, i.e., coefficients in R, can be reduced to the form

$$f = A_0 + A_1\theta + \cdots + A_{n-1}\theta^{n-1},$$

where $A_0, A_1, \cdots, A_{n-1}$ are rational numbers; and the assemblage of all such functions is referred to as the *field* or *Körper* $K(\theta)$. If f is an algebraic integer, the numbers conjugate to f are also algebraic integers, and by writing down the conjugate equations and solving, we find that $d(\theta)A_0$, $d(\theta)A_1$, $\cdots$ are rational integers where $d(\theta)$ is the discriminant of the equation in θ. Hence any algebraic integer in the field $K(\theta)$ can be written as

$$f = \frac{1}{d(\theta)}(a_0 + a_1\theta + \cdots + a_{n-1}\theta^{n-1}),$$

where $a_0, a_1, \cdots, a_{n-1}$ are integers. From this it follows that we can find n algebraic integers $\omega_1, \omega_2, \cdots, \omega_n$ called the base of the field, such that any algebraic integer f can be written in the form

$$f = x_1\omega_1 + x_2\omega_2 + \cdots + x_n\omega_n,$$

where $x_1, x_2, \cdots, x_n$ are rational integers. For example, in the field $K(\sqrt[3]{2})$, all integers are of the form $x_1 + x_2\sqrt[3]{2} + x_3\sqrt[3]{4}$, while in the field $K(\sqrt{5})$ they are of the form $x_1 + x_2(1 + \sqrt{5})/2$.

The base can be chosen in an infinite number of ways, but any one base can be derived from any other by a linear substitution in the ω's with integer coefficients and with a determinant unity. Hence the square of the determinant formed from the ω's and their conjugates, that is,

$$d = \begin{vmatrix} \omega_1^{(1)}, \omega_2^{(1)}, \cdots, \omega_n^{(1)} \\ \omega_1^{(2)}, \omega_2^{(2)}, \cdots, \omega_n^{(2)} \\ \cdots\cdots\cdots \end{vmatrix}^2,$$

where $\omega_1^{(2)}$, etc., are the conjugates of $\omega_1^{(1)} = \omega_1$, etc., is an invariant of the bases, and is a rational integer called the discriminant d of the field. It is always greater than unity, and there are only a finite number of algebraic fields with a given discriminant d, as follows from an asymptotic formula given by Minkowski (as an example of his result stated further on), namely

$$d \sim \frac{1}{2\pi n}\left(\frac{\pi}{4}\right)^{2r_2} e^{2n - \frac{1}{6n}}.$$

3. *Units.* Among algebraic integers the most important are the units, i.e., the divisors of unity. For example, in the field $K(i)$, $\pm 1, \pm i$ are divisors of unity, while in the field $K(\sqrt{2})$, $t + u\sqrt{2}$ is a unit if the integers t, u satisfy the equation $t^2 - 2u^2 = \pm 1$. It is well known that all the units are given by $\pm(1 + \sqrt{2})^n$, where n is any integer.

A similar theory holds in the general case, since it was proved by Dirichlet that any unit can be represented in the form $e_1^p e_2^q, \cdots$ for a finite number of values of $e_1, e_2, \cdots$, where $p, q, \cdots$ are any integers. For example, in the field $K(\sqrt[3]{2})$, the units are of the form $x + y\sqrt[3]{2} + z\sqrt[3]{4}$, where x, y, z are integers satisfying the equation

$$x^3 + 2y^3 + 4z^3 - 6xyz = \pm 1,$$

where the left-hand side is the norm of $x + y\sqrt[3]{2} + z\sqrt[3]{4}$,

so that the theory gives the complete solution of this equation in x, y, z. It has lately been announced that it can be shown in this way that the equation $x^3 - ay^3 = 1$ never has more than one integer solution when a is given.

4. *Minkowski's Theorem.* It is Minkowski's theorem on linear forms, however, which is fundamental in the theory and has contributed greatly to its simplicity and elegance, namely, that integer values of x and y, not both zero, can be found so that

$$|ax + by| \leqq p, \qquad |cx + dy| < q,$$

where a, b, c, d are any real numbers, and $p, q > 0$, satisfy the equation

$$pq = \begin{vmatrix} a, & b \\ c, & d \end{vmatrix}.$$

There is of course the obvious extension to any number of variables $x, y, z, \cdots$.

No less than four distinct proofs have been given. The original proof by Minkowski is equivalent to the geometric theorem that any parallelogram in the x, y plane with one vertex at the origin and area $\geqq 1$ contains at least one lattice point on its sides or within its interior and is really a particular case of a far more general one. This proof, as well as the latest one just given by Siegel, which is analytic in character and depends upon trigonometric series, applies directly regardless of whether the coefficients are rational or not. The other proofs are arithmetic in character, first establishing the theorem for rational coefficients. Hilbert's proof depends upon Dirichlet's idea that if $n + 1$ objects are arranged in n groups, then one group will contain at least two objects. The proof by Hurwitz is a beautiful piece of arithmetic work, very characteristic of the author, showing that there are

$$\begin{vmatrix} a, & b \\ c, & d \end{vmatrix}$$

forms of the type $\lambda x + \mu y$, such that all forms $Ax + By$, with A, B integers, can be written in the form

$$\lambda x + \mu y + p(ax + by) + q(cx + dy),$$

where all coefficients, etc., are integers, and where a, b, c, d are given.

5. *Test for Algebraic Numbers.* I may say here a few words regarding the conditions that a given number θ should be an algebraic number of the nth degree, i.e., the root of an equation of the nth degree irreducible in R. Of the two kinds known, one states that a number θ cannot be an algebraic number of the nth degree if we can find an infinity of rational approximations p/q, such that

$$\left|\frac{p}{q} - \theta\right| < \frac{c}{q^\lambda},$$

where c is a given number, and where $\lambda > n$ according to Liouville, $\lambda \geqq n$ according to Thue in 1908, or finally $\lambda > 2\sqrt{n}$ according to Siegel in 1921. From this flow such results as that the equation $f(x, y) = c$ has only a finite number of integer solutions if f is an irreducible binary quantic in x, y of degree greater than 2. The proofs are very complicated, but very remarkable, depending only on elementary algebra.

The other types of results are due to Minkowski and Furtwängler and depend upon the investigation of the minima of the form

$$x_0 + x_1\theta + \cdots + x_{n-1}\theta^{n-1}$$

for integer values of $x_0, x_1, \cdots, x_{n-1}$, all numerically less than some number t. If now t takes the values $1, 2, \cdots$, we have a series of minima $m_1, m_2, m_3, \cdots$, which are such that the ratios $m_2/m_1, m_3/m_2, \cdots$ have only a finite number of values for all values of t.

6. *Ideals.* The algebraic integers in the field $K(\theta)$ form the foundation of all that follows, just as ordinary rational integers do in arithmetic, and the word *integer* hereafter refers to the integers in the field $K(\theta)$.

Let $\alpha_1, \alpha_2, \alpha_3, \cdots, \alpha_p$ be any given integers; then the *ideal* A is the group of integers defined by $\lambda_1\alpha_1 + \lambda_2\alpha_2 + \cdots$, where $\lambda_1, \lambda_2, \lambda_3, \cdots$ are any integers, a fact expressed by the notation

$$A = [\alpha_1, \alpha_2, \alpha_3, \cdots, \alpha_p].$$

The ideals A and $B = [\beta_1, \beta_2, \beta_3, \cdots, \beta_q]$ are equal, written $A = B$ if every number of A is included in B and conversely; i.e., if integers λ, μ can be found so that

$$\beta_1 = \lambda_{11}\alpha_1 + \lambda_{12}\alpha_2 + \cdots, \qquad \beta_2 = \lambda_{21}\alpha_1 + \lambda_{22}\alpha_2 + \cdots,$$

and

$$\alpha_1 = \mu_{11}\beta_1 + \mu_{12}\beta_2 + \cdots, \qquad \alpha_2 = \mu_{21}\beta_1 + \mu_{22}\beta_2 + \cdots.$$

An ideal is called a *principal ideal* if it can be written in the form $[\alpha]$, so that it consists of all the integers divisible by α. Further the principal ideals $[\alpha]$ and $[\beta]$ are equal if and only if α and β are associated integers, i.e., $\alpha = \beta e$, where e is a unit.

In particular, if $\alpha = 1$, we have as the unit ideal [1] all the integers in $K(\theta)$.

We can now extend many arithmetic concepts to ideals. Thus the product AB of the ideals A and B is defined as the ideal C formed from the numbers obtained by multiplying every number of A by every number of B; and we write $C = AB$. The commutative law is obviously satisfied, so that we can write $A \times A = A^2$, $A \times A \times A = A^3$, etc., while A^0 stands for the unit ideal [1].

Division is defined as the inverse of multiplication, so that the ideal C is divisible by the ideal A if an ideal B can be found so that $C = AB$. An ideal P is called a *prime ideal* if it is divisible by only itself and [1], the unit ideal. Finally, two ideals are called *prime to each other* if they have no common divisor except [1].

The fundamental theorem in the theory of ideals states that an ideal can be factored in only one way, apart of course from the order of the factors. Many important consequences follow just as in elementary number theory. The proof can be presented in several different ways, requiring in any case a long chain of subsidiary propositions. In Hurwitz' method the important steps are as follows:

(1) Corresponding to any ideal A we can find an ideal B so that AB is a principal ideal $[\alpha]$ where α is a positive rational integer.

(2) From the equality $[\gamma]A = [\gamma]B$, we have $A = B$, and hence from $CA = CB$, also $A = B$. From these we show that if C is divisible by A, every number of C is included in A, and conversely.

It is then shown that an ideal A has only a finite number of divisors, since a given rational integer α is a member of only a finite number of ideals.

The next step is to show that if an ideal P is a divisor of AB, then either A or B is divisible by P. This depends upon the fact that the greatest common divisor of the ideals A, B is given by $[\alpha_1,\alpha_2, \cdots, \alpha_p, \beta_1, \beta_2, \cdots, \beta_q]$. The result follows immediately.

An ideal can be factored by a definite and direct process. We can then factor any algebraic integer I by factoring the principal ideal $[I]$. Suppose we find

$$[I] = A^\lambda B^\mu C^\nu \cdots,$$

where $A, B, C, \cdots$ are different prime ideals. Then if $A, B, C, \cdots$ are principal ideals $[\alpha]$, $[\beta]$, $[\gamma]$, $\cdots$, we have the result that

$$I = \epsilon \alpha^\lambda \beta^\mu \gamma^\nu \cdots,$$

where ϵ is a unit. If, however, one at least of the ideals $A, B, C, \cdots$ is not a principal ideal, the integer I cannot be factored in the ordinary sense, though the ideal $[I]$ can be. It is for this reason that I cannot be considered as a prime, and that a unique factorization law that would naturally suggest itself does not hold in the theory of algebraic numbers.

7. *Congruences and Norms.* We can now consider congruences with respect to ideals. A number μ is said to be divisible by the ideal A if μ is one of the numbers forming the ideal A, or, what amounts to the same thing, if the principal ideal $[\mu]$ is divisible by A; and we write

$$\mu \equiv 0 \pmod{A}.$$

Hence we mean by the congruence

$$\mu \equiv \nu \pmod{A}$$

that $[\mu - \nu]$ is divisible by A, or that $\mu - \nu$ belongs to the ideal A. The ideal A is only a part of the integers of $K(\theta)$, and the number of incongruent integers (mod A) is finite and is denoted by $N(A)$. In particular, if A is a principal ideal $[\mu]$, $N(A)$ is the absolute value of the product of μ by its conjugates. For example, in the field $K(i)$, if ω is a prime of the form $a + ib$, where $p = a^2 + b^2$ is a rational prime $\equiv 1 \pmod 4$ and where a, b are rational integers, the number of incongruent integers (mod ω) is

$$(a + ib)(a - ib) = p.$$

If μ is a rational integer, the number $N(\mu)$ becomes $|\mu^n|$.

The norm satisfies the law

$$N(A)N(B) = N(AB).$$

To find a simple formula for the norm of an ideal, we first prove that an ideal has a base, i.e., n integers $\alpha_1, \alpha_2, \cdots, \alpha_n$ can be found so that all the numbers of A can be written in the form

$$\lambda_1\alpha_1 + \lambda_2\alpha_2 + \cdots + \lambda_n\alpha_n,$$

where $\lambda_1, \lambda_2, \cdots, \lambda_n$ are rational integers, and further that the determinant

$$\begin{vmatrix} \alpha_1^{(1)}, \alpha_2^{(1)}, \cdots, \alpha_n^{(1)} \\ \alpha_1^{(2)}, \alpha_2^{(2)}, \cdots, \alpha_n^{(2)} \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \end{vmatrix}$$

formed from the base and its conjugates is an invariant of the ideal. The numbers $\alpha_1, \alpha_2, \cdots, \alpha_n$ of the base can be expressed in terms of the base $\omega_1, \omega_2, \cdots, \omega_n$ of the field by means of the equations

$$\begin{aligned} \alpha_1 &= c_{11}\omega_1 + c_{12}\omega_2 + \cdots, \\ \alpha_2 &= c_{21}\omega_1 + c_{22}\omega_2 + \cdots, \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \end{aligned}$$

where the c's are rational integers. The determinant $|c_{kl}|$ is the norm of the ideal A. We can also write

$$|\alpha_{kl}|^2 = dN^2(A),$$

a most useful equation.

The norm of a prime ideal P takes a very simple form. For P must be a divisor of a rational prime p which is the smallest rational integer divisible by P. Hence we have

$$N(P) = p^f, \quad \text{where} \quad 1 \leqq f \leqq n.$$

The integer f is called the *grade* of the ideal. It should be noted that only a finite number of ideals have a given norm.

There are of course an infinite number of ideals, but they can be divided into a finite number of classes. Thus an ideal A will be called equivalent to B, i.e., they are in the same class, if integers α, β can be found so that

$$[\alpha]A = [\beta]B,$$

a fact expressed by writing $A \sim B$.

All the principal ideals are equivalent, and they constitute the principal class. Further, there are only a finite number of classes, as follows from the fact that every class contains an ideal whose norm $\leqq \sqrt{d}$. This is a simple deduction from the fact, depending upon Minkowski's theorem, that every ideal A contains a number a such that

$$|N(a)| \leqq N(A)\sqrt{d}.$$

We could also in this way actually calculate H, the number of ideal classes. The classes of ideals form an ordinary abelian group.

The whole theory can be developed by proving that H is finite, and then deducing the unique factorization law.

8. *Application to Indeterminate Equations.* This number H, and the fact that it is finite, are of the greatest importance in the applications to indeterminate equations. For if A is any ideal, then A^H belongs to the principal class, i.e., $A^H \sim [1]$. Conversely, if $A^n \sim [1]$, and n is prime to H, it follows that A is a principal ideal. Hence, if we wish to draw any conclusion from the equality $ab = c^n$ in algebraic integers, we must first write it as an equation in ideals $[a][b] = [c]^n$. Therefore, if the ideals $[a]$, $[b]$ have a common factor, say a principal ideal $[d]$, we must have, say,

$$[a] = [d]A^n, \qquad [b] = [d]^{n-1}B^n,$$

where $[c] = AB$; and the ideals A and B will be principal ideals only if n is prime to H. We then have

$$A = [x], \qquad B = [y],$$
$$[a] = [d][x]^n, \qquad [b] = [d]^{n-1}[y]^n,$$

whence $a = dx^n e$, where e is some unit.

For example, in Fermat's equation in rational integers,

$$x^p + y^p = z^p,$$

or

$$(x + y)(x + \zeta y)(x + \zeta^2 y) \cdots = z^p,$$

where $\zeta = e^{2\pi i/p}$, we consider first the case when x, y, z are all prime to p, that is, the greatest common factor of the ideals $[x + \zeta y], [x + \zeta^2 y], \cdots$, is unity. Hence, if p is prime to the number of ideal classes in the field $K(\zeta)$, we have

$$x + \zeta y = \epsilon a^p,$$

where ϵ is a unit and a is an algebraic integer. It is not very difficult to prove from this equation that Fermat's equation is impossible. Similarly, when one of x, y, z is divisible by p, all, of course, on the assumption that p is prime to H.

Another illustration is given by the classical indeterminate equation

$$az^2 = x^4 + bx^3y + cx^2y^2 + dxy^3 + ey^4,$$

where a, b, c, d, e are given rational integers and x, y, z are unknown rational integers. If θ is a root of the equation

$$\theta^4 + b\theta^3 + c\theta^2 + d\theta + e = 0,$$

az^2 has a factor $x - \theta y$, so that we have the equation in ideals

$$[x - \theta y] = \mu\tau^2,$$

where μ is one of a finite number of ideals and τ is an unknown ideal. Since the number of ideal classes is finite we can put $\tau = \alpha\nu/\beta$, where α, β are integers, and ν is one of a finite number of ideals. Then $\mu\nu^2$ must be one of a finite number of principal ideals, say $[\gamma]$, so that

$$[x - \theta y] = \alpha^2[\gamma]/\beta^2,$$

whence

$$x - \theta y = \epsilon\alpha^2\gamma/\beta^2,$$

where ϵ is a unit. Since all units can be expressed in the form $\epsilon_1\epsilon_2^2$ for a finite number of values of ϵ_1, we deduce an equation

of the form

$$x - \theta y = \nu\alpha^2,$$

where ν is one of a finite number of integers and α is an unknown integer. If this has an infinite number of solutions, we have for a particular solution

$$x_0 - \theta y_0 = \nu\alpha_0^2,$$

whence, by multiplication, we can deduce an equation of the form

$$(x - \theta y)(x_0 - \theta y_0) = M^2(a + b\theta + c\theta^2 + d\theta^3),$$

where x, y, a, b, c, d are unknown rational integers and x_0, y_0, M are given rational integers. It was from an equation of this form (e.g., with $x_0 = 1$, $y_0 = 0$) that I showed that the method of infinite descent applied to the original equation, and hence to the homogeneous ternary cubic

$$f(x, y, z) = 0,$$

i.e., that all its rational solutions could be derived from a finite number by the classic method.*

9. *The Class Number.* The problem of finding the number H of ideal classes is a very interesting and difficult one. Analytically H is a multiple of the residue at $s = 1$ of the function defined when the real part of s is greater than one, by

$$f(s) = \sum \frac{1}{N[A]^s},$$

where the summation refers to all the ideals of the field $K(\theta)$. This is deduced from the series

$$f_L(s) = \sum_A \frac{1}{N(A)^s},$$

the summation referring to all the ideals in the class L, and the residue at $s = 1$ being independent of the class L. We may also write

$$f_L(s) = N(B)^s \sum_\mu \frac{1}{N(\mu)^s},$$

where the summation refers to all non-associated integers μ divisible by the ideal B, and B is an ideal in the class L^{-1}.

*The same method shows that $Ey^3 = Ax^3 + Bx^2 + Cx + D$ has only a finite number of integer solutions if the right side has no squared factor in x.

The function $f(s)$ is the analog of the ordinary Riemann zeta function. Its chief properties remained unknown for many years, and their investigation was one of the problems proposed by Hilbert in his address in 1900 to the International Mathematical Congress at Paris.

In 1916, it was shown by Hecke that $f(s)$ represented a function of s which can be continued throughout the s plane, whose only singularity is a simple pole at $s = 1$, and which also satisfies a very simple functional relation. His method can be illustrated by considering the ordinary ζ function

$$\zeta(s) = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \cdots.$$

By using the gamma function, we can express this as a definite integral

$$\int_0^\infty \xi^{\frac{s}{2}-1}\theta(\xi)d\xi,$$

where $\theta(\xi)$ is practically a theta function. The range of integration is split into $\int_1^\xi + \int_0^1$. In the former, ξ is changed into $1/\xi$ and the classical transformation formula for $\theta(1/\xi)$ is applied. The result at once follows. Moreover, a simple functional equation between $\zeta(s)$ and $\zeta(1 - s)$ is apparent.

The same method applies to $f_L(s)$ by noting that

$$N(\mu) = \mu_1\mu_2 \cdots \mu_n,$$

so that $f_L(s)$ can be transformed into a multiple integral with limits ∞, 0 by writing

$$\Gamma_{(s)}k^{-s} = \int_0^\infty e^{-kt}t^{s-1}dt$$

if k is real and positive, $R(s) > 0$. The great difficulty was to express the fact that the summation refers to the non-associated integer μs, i.e., that only one number of the group $e_1^p e_2^q \cdots \mu$ arising from any integer values of p, q appears in the summation. By writing the limits of integration as

$$\int_0^1 + \int_1^2 + \int_2^3 + \cdots,$$

Hecke transformed the integral into a theta series with n vari-

ables associated with the ideal, where the summation now refers to all the integers μ (not merely the non-associated one). Moreover, since the multiple theta series also had a simple transformation formula, he was able to find a simple functional relation between $f(s)$ and $f(1-s)$, to show that $f(s)$ exists all over the s plane, and has a simple pole at $s=1$. Siegel has lately shown that these results can be found in another way, wherein the units make no appearance. The same method applies to many functions associated with or derived from the Dedekind zeta function, such as the series

$$\sum_A \frac{\chi(A)}{[N(A)]^s},$$

where now $\chi(A)$ is a root of unity associated with the ideal A, and many results in the ordinary theory of prime numbers can be extended to the case of prime ideals. For example, the prime number theorem, which states that the number of primes less than x is asymptotically equal to $x/\log x$, is equivalent to the fact that the Riemann zeta function has only a simple pole at $s=1$ and no zeros in a contour whose right-hand boundary is say $z=2$, and whose left-hand boundary approaches $z=1$ from the left according to the law

$$y = 1 - \frac{a}{\log x},$$

where a is a constant. The Dedekind zeta function has practically the same properties as the Riemann function whence results the same asymptotic formula for ideals as for ordinary primes, e.g., the number of prime ideals whose norm $\leqq x$ is asymptotically equal to $x/\log x$. Further Hecke was able to prove results such as that the indefinite form $ax^2+bxy+cy^2$ represents an infinite number of primes in any given sector of the xy plane whose center is at 0 and whose radius is infinite. The θ functions dealt with by Hecke are associated with ideals and algebraic numbers in a very simple and elegant manner. A simple case is that in which we have

$$\theta(t,A) = \sum_{\mu=0 \pmod A} e^{-c|\mu_1|^2 t_1 - c|\mu_2|^2 t_2 - \cdots - c|\mu_n|^2 t_n},$$

where the summation refers to all the numbers $\mu = \mu_1$ of an ideal A, and $\mu_2, \mu_3, \cdots$ are the conjugates of μ, and c is a constant depending on the ideal A. Then we may write

$$\theta(t, A) = \frac{1}{\sqrt{t_1 t_2 \cdots t_n}} \theta\left(\frac{1}{t}, B\right),$$

where B is an ideal derived very simply from A, namely $AB = 1/D$, where D is the *grund* ideal, really an ideal whose norm is the discriminant d.

These θ functions led Hecke to the consideration of Gauss' sums in any algebraic field, for which they are as important as are the ordinary Gauss' sums in the elementary theory of numbers. He showed that the reciprocity formula for them follows from the functional equation for the zeta function, and that the law of quadratic reciprocity for any algebraic field is a simple consequence of his general methods. His functional equations have enabled him to prove a number of striking results, both arithmetic and analytic. One of the latter is that if $R(x)$ is the fractional part of x so that $0 \leqq R(x) < 1$, while $x - R(x)$ is an ordinary integer, then the Dirichlet series

$$\sum_{m=1}^{\infty} \frac{R(ma) - 1/2}{m^s},$$

where $a = \sqrt{d}$ or $1/\sqrt{d}$, represents a meromorphic function of s, analytic for $R(s) > 0$, while if $R(s) \leqq 0$, it has simple poles at the points represented by the formula

$$s = -2k \pm \frac{2n\pi i}{\log \eta}, \quad (n, k = 0, 1, 2, \cdots),$$

where η is the fundamental unit or its square in the field $K(\sqrt{d})$.

10. *Representations as Sums of Squares.* Siegel has shown that the methods introduced by Hardy and Littlewood into the analytic theory of numbers can also be extended to similar questions involving algebraic numbers. Thus the question of finding approximate formulas for the number of representations of a given rational integer n as a sum of say 5 squares is equivalent to finding the coefficient of x^n in the expansion

of $f(x) = (1 + 2x + 2x^4 + \cdots)^5$. This is given by a contour integral around a circle whose radius is taken to be very nearly one, say $1 - 1/n$. The range of integration 2π and 0, or say 1, 0, is split up into arcs according to the Farey method of division of order, say $[\sqrt{n}]$. If the fraction p/q is associated with one of these arcs, the integrand is evaluated approximately at $x = e^{2\pi i p/q}$ and the sum of the resulting integrals is proved to be a genuine approximation. Siegel has shown that if we seek the number of representations of an algebraic integer as a sum of squares of algebraic integers (when the field and its conjugates are real), the θ function $1 + 2x + 2x^4 + \cdots$ can be replaced by the θ series considered by Hecke. For example, in a quadratic field, the coefficient involves a double integral over the unit square. This square can be subdivided in a method similar to the ordinary Farey method into a number of small regions, and in each of these regions an approximate value is taken for the integrand, and the resulting integral again gives a genuine approximation.

11. *Laws of Reciprocity.* Finally the general laws of reciprocity in any field, i.e., the investigation of the congruence

$$x^n \equiv p \pmod{Q},$$

where Q is a given ideal, p a given algebraic integer, x an unknown algebraic integer, and n a given rational integer, are some of the most successful, abstruse, and far reaching results of the ideal theory, giving one a glimpse of regions so remote that apparently many years will elapse before our efforts will bring us within measurable distance.

First consider the congruence $x^2 \equiv q \pmod{p}$ in rational numbers, where p and q are odd primes. As factorization of $x^2 - q$ suggests $\sqrt{q}$, let us examine the meaning of this congruence in the field $K(\sqrt{q})$. If it is possible, it is equivalent to saying that the prime p factors in the field $K(\sqrt{q})$. By associating with any ideal a certain number of roots of unity, say $e_1, e_2, \cdots, e_k$, we can divide the ideal classes into genera; and conversely, if these units are given and satisfy an equation of consistency $e_1 e_2 \cdots e_k = 1$, we can find a class of ideals

associated with the given units. Not only can we prove the law of quadratic reciprocity in this way, as was done by Kummer, but the ideas involved are so general that they have been extended by Hilbert and Furtwängler to the law of reciprocity in any field. Many new ideas, however, are involved in the proof, e.g., if $x^2 \equiv q \pmod{p}$ where now x, q are integers, and p is an ideal in the field $K(\theta)$, we have to investigate the properties of the algebraic field $K(\sqrt{q})$. It is of course obvious that $\sqrt{q}$ satisfies an equation of degree $2n$, but it is more convenient to consider it as a number satisfying a quadratic equation, the coefficients being integers in $K(\theta)$, so that $\sqrt{q}$ generates a quadratic field relative to the field $K(\theta)$. So it is more convenient to consider the ideals in the field $K(\sqrt{q})$ as quadratic ideals relative to the field $K(\theta)$. If the field $K(\theta)$ satisfies certain very special conditions, the study of the relative field $\sqrt{q}$ leads to the law of quadratic reciprocity in the field. All the laws of reciprocity, quadratic, cubic, etc., can be deduced when the proper relative field is known. This is a question of great difficulty and importance.

Great progress has been made arithmetically as far as the laws of reciprocity are concerned. In particular, the relative discriminant of these fields is unity and the relative Galois group is isomorphic with the group of the ideal classes in $K(\theta)$. Analytically, however, it has only been done in a few cases. The problem is equivalent to questions such as the following. Take the equation $x^p = 1$ where p is a prime. This is an abelian equation, i.e., all its roots are rationally represented in terms of one of them, e.g., ζ, ζ^2, ζ^3, ζ^4, $\cdots$, or, say, $f_1(\zeta)$, $f_2(\zeta), f_3(\zeta), \cdots$, and it is obvious that

$$f_a[f_b(\zeta)] = f_b[f_a(\zeta)].$$

Then there is a theorem which states that the root of any abelian equation whose coefficients are rational integers, can be expressed rationally with ordinary rational coefficients in terms of roots of unity. The next stage is that the roots of any abelian equation whose coefficients are imaginary quadratic integers in the field $K(\omega)$ can be expressed rationally in

terms of $j(\omega)$, where $j(\omega)$ is the well known modular function, that is, by means of the complex multiplication of elliptic functions.

Beyond this, however, we at present cannot go. But it is obvious what a field of research is suggested for the future.

Some of the relative fields are given by the equations in elliptic functions dealing with the subdivision of the periods, and also by the modular functions of several variables, an idea due to Hilbert, and developed by Blumenthal and Hecke and intimately connected with the θ function in the Riemann theory of algebraic functions.

The final law of reciprocity can be stated in all its generality in a remarkably simple form. For example, the law of quadratic reciprocity in any field $K(\theta)$ is equivalent to the theorem that the equation

$$ax^2 + by^2 + cz^2 = 0,$$

where a, b, c are given and x, y, z are unknown coprime integers in the field, is possible if and only if the congruence

$$ax^2 + by^2 + cz^2 \equiv 0 \pmod{P}$$

can be satisfied if P is any ideal in the field. A simple proof of this would of course lead to an easy arithmetical proof of the laws of reciprocity, and it is well worth the attention of mathematicians.

Finally, we may state that Siegel has made recently an interesting application of the law by showing that every algebraic integer can be expressed as the sum of 4 squares of algebraic numbers provided it is totally positive, that is, those of the conjugates that are real must be positive. Waring's theorem also has been extended to algebraic numbers.*

THE UNIVERSITY OF MANCHESTER, ENGLAND

* For references on this and other topics mentioned above, see the *Report on Algebraic Numbers*, BULLETIN OF THE NATIONAL RESEARCH COUNCIL, February, 1923; H. Bohr and H. Cramer, *Die neuere Entwickelung der analytischen Zahlentheorie*, ENCYKLOPÄDIE, II C 8; Hecke, *Vorlesungen über die Theorie der Algebraischen Zahlen*; Mordell, *Indeterminate equations of the third degree*, SCIENCE PROGRESS, July, 1923.

INTEGRAL SOLUTIONS OF $x^2 - my^2 = zw$ *

BY L. E. DICKSON

1. *Statement of the Theorem.* In this BULLETIN (vol. 27 (1920–1), p. 361) I stated that all integral solutions of the equation

$$x^2 - my^2 = zw$$

are obtained by multiplying the right members of

$$(1) \quad z = el^2 + 2flq + gq^2, \qquad w = en^2 - 2fnr + gr^2,$$

$$(2) \quad x = \pm (eln + fnq - flr - gqr), \qquad y = lr + nq,$$

by the same arbitrary integer, where e, f, g take only those sets of integral values (finite in number) for which the first form (1) is a reduced quadratic form having the same discriminant $4m$ as $x^2 - my^2$, so that

$$(3) \qquad f^2 - eg = m.$$

In other words, we employ a single form $el^2 + \cdots$ from each class of quadratic forms of discriminant $4m$. The number of such classes is therefore the number of sets of formulas (1), (2) giving all integral solutions of $x^2 - my^2 = zw$. If we permit the interchange of z and w, we need retain only a single sign in (2). For, if we replace l, q, n, r by $n, -r, -l, q$, respectively, we find that z and w are interchanged, y is unaltered, and x is replaced by $-x$.

In the paper cited, I was led to the above theorem by the theory of ideals, and I gave a proof when there is a single class of quadratic forms.† I stated that a simpler proof of the general theorem follows by composition. I have since found the following still simpler proof.

2. *A Simplification.* Since we may lay aside a common factor of x, y, z, w, consider $X^2 - mY^2 = ZW$, where X, Y, Z, W

* Presented to the Society, December 1, 1923.

† A complete proof of the general theorem by means of ideals has been found recently by G. E. Wahlin.

are integers without a common factor greater than 1. We may write

$$X = dx, \qquad Y = dy, \qquad d = \delta D, \qquad Z = \delta z_1,$$

where x and y are relatively prime, and likewise D and z_1, while δ is prime to W. Then $d^2(x^2 - my^2) = \delta z_1 W$, so that $z_1 W$ is divisible by δD^2. Hence $z_1 = \delta z$, $W = D^2 w$. We have been led back to our initial equation $x^2 - my^2 = zw$ with the simplification that x and y are now relatively prime.

3. *Argument.* If a prime divided both y and w, it would divide x^2 and hence also x, contrary to the last result. Since $y = l$ and $w = g$ are therefore relatively prime, there exist integers f and q such that $x = fl + qg$. Inserting these values of x, y and w in $x^2 - my^2 = zw$, we get

$$zg = (f^2 - m)l^2 + 2fglq + g^2q^2. \tag{4}$$

Since g is prime to l, we have (3), where e is an integer. Cancellation of g gives (1_1).

Let the substitution, with integral coefficients,

$$l = rl_1 + nq_1, \qquad q = sl_1 + tq_1, \qquad rt - ns = 1,$$

replace the first form (1) by a reduced form

$$F(l_1, q_1) = e_1 l_1^2 + 2f_1 l_1 q_1 + g_1 q_1^2.$$

The latter is therefore replaced by the former by the inverse substitution

$$l_1 = tl - nq, \qquad q_1 = -sl + rq.$$

Taking $l = 0$ for the moment, we see that the coefficient $g = w$ of q^2 in (1_1) is

$$F(-n, r) = e_1 n^2 - 2f_1 nr + g_1 r^2.$$

Moreover, we have $y = l = rl_1 + nq_1$. Dropping the subscripts, we have the expressions for z, w, y in (1), (2). While the expression (2) for x may be derived from $x^2 - my^2 = zw$ by direct computation, it may be obtained more simply as follows. We have

$$gz = (fl + qg)^2 - ml^2, \qquad gw = (gr - fn)^2 - mn^2, \tag{5}$$

the first being another form of (4). The product of their linear factors is

$$(6) \quad (fl + qg + l\sqrt{m})(gr - fn + n\sqrt{m}) = -g(\pm x - y\sqrt{m}),$$

in view of (2). Taking the product of (6) by the identity derived from it by changing the sign before $\sqrt{m}$, and removing the factor g^2, we get $zw = x^2 - my^2$. Hence x has the values in (2). This proves that conversely (1) and (2) are solutions for all values of l, q, n, r.

4. *Conclusion.* It follows that every set of integral solutions of $x^2 - my^2 = zw$ (with x and y relatively prime) is obtained by multiplying the second members of (1), (2) by the same arbitrary integer. The restriction in parenthesis may be discarded. For, by §2, the general solution of $X^2 - mY^2 = ZW$ in integers is given by

$$X = \delta Dx, \qquad Y = \delta Dy, \qquad Z = \delta^2 z, \qquad W = D^2 w.$$

Let us write

$$L = \delta l, \qquad Q = \delta q, \qquad N = Dn, \qquad R = Dr.$$

Then the values of Z, W, X, Y are derived from (1), (2) by replacing l, q, n, r by L, Q, N, R, respectively.

5. *A Companion Theorem.* The above proof applies without essential change to the solutions of the equation

$$(7) \quad x^2 + xy + \tfrac{1}{4}(1 - m)y^2 = zw, \qquad m \equiv 1 \pmod 4,$$

and hence leads to the companion theorem of the former paper.

6. *Generalization.* As a generalization, consider

$$(8) \qquad ax^2 + 2bxy + cy^2 = zw.$$

Multiply by a and write $m = b^2 - ac$, $\xi = ax + by$. Then

$$(9) \qquad \xi^2 - my^2 = z \cdot aw.$$

Hence z, aw, ξ, y are given by the products of the right members of (1), (2) by the same arbitrary integer. The second form (1) will be divisible by a prime factor p of a if and only if n and r satisfy two linear homogeneous congruences modulo p (which may coincide); they serve to express n and r in one or two ways as linear homogeneous functions of new parameters

N and R. The initial form (1_2) becomes one or two new quadratic forms in N and R. We proceed similarly with a prime factor of a/p, etc. Finally, we obtain formulas for x from $ax = \xi - by$. We conclude that all integral solutions of (8) are products of the same arbitrary integer by the numbers obtained from a finite number of sets of four expressions each quadratic in four arbitrary parameters. The explicit formulas will be discussed on another occasion.

THE UNIVERSITY OF CHICAGO

ON THE REALITY OF THE ZEROS OF A λ-DETERMINANT *

BY R. G. D. RICHARDSON

Some of the best-known theorems of algebra are centered around the zeros of the polynomial in λ,

$$(1) \qquad \begin{vmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & \cdots & a_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{nn}-\lambda \end{vmatrix}.$$

In the classical case of the determinant connected with the equations of secular variations, where the elements a_{ij} are real and the determinant $|a_{ij}|$ formed from (1) by omitting the λ's is symmetric ($a_{ij} = a_{ji}$), these zeros turn out to be real. This theorem concerning the reality of the zeros has been extended † to the case where a_{ij} and a_{ji} are conjugate complex ($a_{ij} = \bar{a}_{ji}$). It is proposed in this note to extend it to a still more general case which has arisen in some investigations concerning pairs of bilinear forms just completed by the author. This generalization consists in allowing the coefficients of the λ's to be n^2 in number instead of n as in (1), of allowing them to be various and complex instead of all unity, and of bordering the determinant by m rows and m columns. The

* Presented to the Society October 27, 1923.

† Cf. Kowalewski, *Einführung in die Determinantentheorie*, p. 130.

method of proof is extremely elegant and it is worthy of notice that for the special case of the known theorems concerning (1), the proof itself is much simpler than any previously given.

Let us consider the determinant*

$$(2)\qquad \begin{vmatrix} a_{11}-\lambda b_{11} & \cdots & a_{1n}-\lambda b_{1n} & a_{1,n+1} & \cdots & a_{1,n+m} \\ \cdot & \cdots & \cdot & \cdot & \cdots & \cdot \\ a_{n1}-\lambda b_{n1} & \cdots & a_{nn}-\lambda b_{nn} & a_{n,n+1} & \cdots & a_{n,n+m} \\ a_{n+1,1} & \cdots & a_{n+1,n} & 0 & \cdots & 0 \\ \cdot & \cdots & \cdot & \cdot & \cdots & \cdot \\ a_{n+m,1} & \cdots & a_{n+m,n} & 0 & \cdots & 0 \end{vmatrix},$$

under the following hypotheses:

(*a*) the elements a_{ij} are conjugate complex;

(*b*) the elements b_{ij} are conjugate complex and the corresponding hermitian form

$$\sum_{ij}^{1,n} b_{ij}z_i\bar{z}_j$$

is definite.†

Under these hypotheses it is proposed to show that the zeros of the $(n-m)$th degree polynomial (2) in λ are real.

PROOF. Consider the $n+m$ homogeneous equations

$$(3)\qquad \begin{cases} \sum_{j}^{1,n}(a_{ij}-\lambda b_{ij})x_j + \sum_{k}^{n+1,n+m} a_{ik}x_k = 0 & (i=1,\cdots,n), \\ \sum_{k}^{1,n} a_{ik}x_k = 0 & (i=n+1,\cdots,n+m), \end{cases}$$

in $n+m$ variables x, whose coefficients are the elements of the determinant (2). The necessary and sufficient condition that

* It is obvious that m must be less than n; otherwise the polynomial would be identically zero, and any theorem concerning the roots necessarily trivial.

† A necessary and sufficient condition that it be positive definite is that the terms of the sequence of $n+1$ determinants

$$1,\ b_{11},\ \begin{vmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{vmatrix},\ \begin{vmatrix} b_{11} & \cdots & b_{13} \\ \cdot & \cdots & \cdot \\ b_{31} & \cdots & b_{33} \end{vmatrix},\ \cdots,\ \begin{vmatrix} b_{11} & \cdots & b_{1n} \\ \cdot & \cdots & \cdot \\ b_{n1} & \cdots & b_{nn} \end{vmatrix}$$

all have the same sign; the condition that it be negative definite is that these terms alternate in sign.

(3) have a solution is that (2) vanish. Hence for any zero of (2), the equations (3) have a solution $x_1, x_2, \cdots, x_{n+m}$, not identically zero. Multiplying the ith equation (3) by $\bar{x}_i$ $(i = 1, \cdots, n+m)$, and adding, we find a relation which may be written in the form

$$\sum_{ij}^{1,n} a_{ij}x_j\bar{x}_i - \lambda\sum_{ij}^{1,n} b_{ij}x_j\bar{x}_i + \sum_{i}^{1,n}\sum_{k}^{n+1,n+m} a_{ik}x_k\bar{x}_i + \sum_{i}^{n+1,n+m}\sum_{k}^{1,n} a_{ik}x_k\bar{x}_i = 0. \tag{4}$$

Now, by hypothesis (a), the terms $a_{ik}x_k\bar{x}_i$, $a_{ki}\bar{x}_kx_i$ are conjugate and their sum is real. Hence the first and second sums in relation (4) are hermitian forms and necessarily real, each of the n^2 terms in each sum being matched by its conjugate in the same sum. Further, since

$$\sum_{i}^{n+1,n+m}\sum_{k}^{1,n} a_{ik}x_k\bar{x}_i \equiv \sum_{k}^{n+1,n+m}\sum_{i}^{1,n} a_{ki}\bar{x}_kx_i,$$

it follows that each of the nm terms in the third sum is matched with a conjugate among the nm terms in the fourth and hence that the sum of these two sums is real. Finally by hypothesis (b), the second sum is different from zero (being zero only when all the x's vanish). Hence λ is real.

THEOREM. *The zeros of the* λ*-determinant* (2) *are real provided hypotheses* (a) *and* (b) *are satisfied.*

In the special case of the determinant of secular variations, (3) and (4) respectively take the forms

$$\sum_{j}^{1,n} a_{ij}x_j = \lambda x_i, \quad (i = 1, \cdots, n); \qquad \sum_{ij}^{1,n} a_{ij}x_ix_j = \lambda\sum_{i}^{1,n} x_i^2.$$

The proof of the reality of the λ's is thus notably simple.

BROWN UNIVERSITY

BOUTROUX ON MATHEMATICAL IDEALS

L'Idéal Scientifique des Mathématiciens dans l'Antiquité et dans les Temps Modernes. By Pierre Boutroux. Paris, Felix Alcan, 1920. 274 pp.

What conception do mathematicians have of their science? What plan do they follow in their research? What principles direct their activity? What is the goal which they seek? These are the questions which the author of this book sets himself and to which he seeks an answer. Few probably will deny the value and importance of such questions nor the desirability that the devotees of mathematics should ask them of themselves and that they should have some clear notions as to the answers. In the vast complexity of modern mathematics, it is surely highly desirable that the man who is working in some particular corner and possibly on a very restricted problem should occasionally pause in his work in order to gain in proper perspective a view of the whole. Whither is he tending? What is his object? What constitutes real progress in mathematical investigations? What is important and why?

Although everyone will probably admit the importance of such considerations, it is to be feared that relatively few have seriously set themselves these questions. Still fewer perhaps have been able to formulate satisfactory replies. Indeed the present work shows how very difficult it is to answer the questions proposed.

The author sets himself a very definite problem. While his inquiry is in the nature of the case philosophical in the broad sense of the term, he is not at all concerned with the question of what place mathematics has in a general system of philosophy. He is concerned merely with the philosophy of mathematics as such. Nor is he concerned at all with the metaphysical aspects of the problem, but purely and simply with answers to the questions proposed. Furthermore he seeks an objective answer. He would eliminate as far as possible all personal bias and seek to discover answers to his questions in the actual work and progress of mathematics itself. His method therefore is, as he says, historical and critical. By a careful examination of the development of mathematics through the centuries he seeks to discover what the leading tendencies were and are. In a word, as indicated in the title of the work, what scientific ideals mathematicians of the past and present have set themselves. He is well aware of the difficulties of his problem and indeed there are many places in the book where the reader will be inclined to differ from the author in his placing of emphasis on such matters. There can be no doubt, however, that the author has written a very stimulating book which may be highly recommended to everyone interested in the questions discussed and should prove of special value to the young investigator starting upon his career and seeking orientation in his chosen field.

The author distinguishes three great epochs in the development of math-

ematics as bearing upon his problem: the epoch of Greek mathematics, the epoch beginning near the end of the 17th century and continuing for 150 years thereafter, and the present epoch beginning about the middle of the last century. The ideals governing the ancient Greeks were very largely æsthetic in character. They dealt with ideal concepts which had no concrete reality. The beauties they sought they considered to be inherent in the objects of study and not to be added to or subtracted from by the human intellect. Moreover, to satisfy their sense of the beautiful, their results had to be simple, harmonious. Another great ideal with which they furnished posterity was that of the geometric demonstration and of the deductive logical system. These ideals governed the development of mathematics through several centuries. They had inherent limitations, however, which the author analyzes with care. A geometric entity did not "exist" for them unless it could be "constructed," and quite obviously their ideals of simplicity and of the rôle that intuition should play in their discoveries involved serious limitations. Herein as well as in their contempt for practical applications may be sought the reasons why the Greeks never developed an algebra.

Preparation for the second epoch came through the introduction of algebra in the middle ages. In direct contrast to Greek ideals, its origin is to be found in practical applications. Algebra came to Western Europe in the form of practical rules for computation with practically no scientific foundation. Moreover progress depended largely on the absence of scientific scruples. The faith of the early investigators was superb. All through the 17th century and beyond, they were governed by the hope that they held in algebraic methods the mechanical key to all science. The new epoch as such may be said to begin with the publication of Descartes' *Geometry*, and with the invention of the calculus by Newton and Leibnitz. Descartes introduced a new conception in place of the euclidean demonstration, a new method which involved the discovery of geometric properties by indirection; that is, by the application of algebraic methods. The new conception of the character of mathematics introduced by the work of Descartes, Newton, and Leibnitz is the idea of synthesis, the idea of putting together simple elements in such a manner as to form progressively compounds of a more and more complicated character. Algebra is considered not as a collection of results but as a method of combination and discussion. In glancing back over the activities of the 150 years following the invention of the calculus, we can readily grasp the enthusiasm of the man working with his new tools. It is hardly an exaggeration to say that mathematical investigation at that time ceased to be a profession and became an industry. No limits were seen to the power of the new methods and all that seemed to be necessary was to proceed systematically in the building up of the edifice from its simple elements to more and more complicated and extensive structures. Leibnitz's dream of a general combinatory calculus whereby all problems of human thought should be capable of solution by an appropriate operational symbolism was a not unnatural consequence of the situation in which he found himself. This era of synthesis is then governed by the ideal that the perfect mathematical science is con-

structive and mechanical, the calculations of which are performed, so to speak, automatically.

The course of events, however, proved the limitations of the new methods. Blocked in its triumphant progress, it is only natural that a critical spirit should develop. The study of the logical foundations of the science became prominent. The author interprets even this phase of activity as belonging in part to the era of synthesis, the idea being that the axioms and postulates sought for were merely in order to provide a secure foundation for the edifice which had been and was being built. It would seem, however, to the reviewer, at least, that the development of postulational methods, especially in their latter course, belongs to the third epoch rather than to the second not merely chronologically, but also in spirit.

This third epoch is again sharply contrasted with the preceding in that it is characterized by analysis rather than by synthesis. The modern mathematician is like a chemist who analyzes an extremely complicated situation and seeks the elements of which it is compounded.* Our present epoch then is characterized by a frank recognition of the limitations of logic alone. Other intellectual activity than that of mere logic is necessary for further progress. Such activities are especially experimentation, the careful analysis of special cases, and above all the recognition of the power of intuition or insight. The modern mathematician must be constructive in the domain of ideas, not merely in the mechanical putting together of simple elements already existing. Progress at present demands the development of new points of view for classifying and interpreting the baffling new problems which present themselves.

In his final chapter on the present mission of mathematics, the author attempts to appraise the manifold and apparently conflicting tendencies that are at present in existence. An extended and interesting discussion of the relation of mathematics to theoretical physics leads to the rather obvious conclusion that the demands of the applications of mathematics cannot furnish the sole or even the principal guide to further progress. The author then takes up the claims of those who would find the desired guide post in the æsthetic or artistic element in mathematics, only to reject this also. He admits that this orientation of our science has been fruitful in that it has served to introduce a large number of new ideas. But, he says, it merely raises the fundamental question in another form: "What precisely does the mathematician mean by 'beautiful,' 'elegant,' 'remarkable?'"

The author thus admits himself unable to give any satisfactory answer

* It may be desirable at this point to caution the reader against a possible confusion of terms. The text of our review would seem to make sufficiently clear the sense in which we (and the author) are using the words "synthesis" and "analysis." The possible confusion arises from the fact that the method used in the second epoch and which would seem to be essentially synthetic in character has received the name of "mathematical analysis."

to the questions he has set himself. He has recourse finally to the advice often given to the young aspirant toward mathematical research: Study the great masters! They had a certain flair for recognizing the valuable and important directions of advance. And the fact remains that, even though we cannot find any simple rules which govern the directions of progress, progress does exist. Our science has advanced and is continually advancing in spite of the lack of any conscious direction.

The author obviously laid down his pen after writing the last word of his interesting little book with a feeling of discouragement and dissatisfaction. The reader shares this feeling;—but, in spite of it, he feels that the writing and the reading has been worth while. The questions raised are of fundamental importance and of the greatest interest. The fact that they remain to a large extent unanswered is merely a challenge to the future. The reviewer has a feeling that the answer may possibly be found in a more vigorous attack on the question which the author himself raises but which he dismisses with a few words. Just what is implied by the words "beautiful," "elegant," "remarkable" as used by the mathematician? Just what is the "flair" which the great masters possess? Is it not possible that this flair is essentially artistic in its nature and that the development of mathematical science is governed largely by laws analogous to those that govern the development of the fine arts?

J. W. YOUNG

SHORTER NOTICES

Ueber Spiralen. By Archimedes. Translated and annotated by Arthur Czwalina-Allenstein. Leipzig, Akademische Verlagsgesellschaft, 1922. 71 pp.

This German translation of Archimedes's classic work on spirals (Ostwald's Klassiker, No. 201), which is now published on account of the fact that Nizze's German translation of 1824 has long been out of print, is of no significance for the American student, as we have Heath's admirable translation.* The supplement (pp. 61–71) gives a reconstruction of a possible method by which Archimedes may have been led to his results; the method is ingenious and plausible, but it has the serious defect that it considers the ratio of an area to a volume, which would have been anathema to a Greek of the classical period; so that we can hardly be convinced, in the absence of evidence, that even so original a genius as Archimedes would have hit upon this particular method. It is probably as well to confess that we are entirely ignorant of the way in which Archimedes did arrive at his admirable results.

R. B. MCCLENON

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* *The Works of Archimedes*, edited in modern notation with introductory chapters by T. L. Heath. Cambridge, 1897.

Einführung in die Ebenen und Körperlichen Oerter. By Pierre de Fermat. Translated and edited by H. Wieleitner. Leipzig, Akademische Verlagsgesellschaft, 1923. 22 pp.

This little volume of the well known Ostwald's Klassiker series is even smaller than the pagination would indicate, as the preface occupies pages 5–6, the translation of Fermat's memoir pages 7–17, and the notes pages 18–22. Fermat, like Descartes and his other contemporaries, follows the Greek nomenclature, "plane" loci meaning straight lines and circles, and "solid" loci meaning conics. Curves of higher degree were called "linear." In this essay the various forms which equations of the first or second degree in two variables can take are briefly discussed, and it is shown geometrically that such equations lead to either "plane" or "solid" loci. Besides its interest to the student of the history of mathematics, the volume may be recommended to undergraduates who are working with the general equation of the second degree, as a striking illustration of the gain in power which they obtain through the use of analytic methods. It seems to the reviewer that some of Fermat's tangent constructions and problems in maxima and minima should have been included, thus rounding out one important section of Fermat's mathematical work, and incidentally bringing the book up to a size comparable with the others of the series.

R. B. McClenon

Theorie und Anwendung der Unendlichen Reihen. By K. Knopp. Berlin, J. Springer, 1922. 10 + 474 pp.

Occasionally one finds an individual who has the ability to present mathematical ideas in lectures in a perfectly translucent elegant manner. It is a real pleasure to listen to such a man, and to watch him develop the subject in a skilful way. It is an ideal towards which every one who lectures on mathematics aspires, but which seems so difficult of attainment. Of the same degree of rarity is the man who has the ability to write as though he were presenting the subject matter to a group of individuals, and who does it in such a delightfully clear way that it is a real joy to read and to absorb. Such seems to be the gift of the author of this volume on infinite series.

The book is not intended to give a profound discussion of the subject. That has recently been done by Pringsheim. Its purpose seems rather to be to provide an introduction to some of the fundamental ideas of the theory of functions of a real variable by showing their application in the theory of limits and particularly infinite series and products. The gap between a course in the calculus, and a profound discussion of the theory of functions of a real variable is often very hard to bridge. The study of infinite series in the manner indicated in this book would be an admirable way of doing this very thing.

A brief survey of the contents will indicate the scope of the book. The first part is devoted to a discussion of the irrational number, leading to a definition of limit of a sequence, the fundamental operations and properties of rational number being assumed. Then follows an elementary discussion of convergence of infinite series, leading to a treatment of power

series, and the expansions of the elementary functions in power series. Infinite products are treated in the same elementary way. Next comes a more extensive study of infinite series, containing a brief survey of the present situation as far as the convergence and divergence of series is concerned. A chapter on series of functions leads naturally to a brief discussion of Fourier series. Similarly, the treatment of series of complex numbers leads to the mention of Dirichlet and factorial series. The book closes with a chapter on divergent series. Problems are provided at the close of each chapter, to lead the reader into a deeper consideration of the subject.

There is much of value in this book for the person who wants an introduction to the study of infinite series, and the person who wants a survey and derivation of much scattered material of interest in this field. Altogether it is a book very much to be recommended to the person who is making his acquaintance with higher mathematics along analysis lines.

T. H. HILDEBRANDT

Early Science in Oxford. Part II. *Mathematics.* By R. T. Gunther. London, Oxford University Press, 1922. 101 pp., 10 plates, colored frontispiece.

The early history of American mathematics is most intimately linked with the development of mathematical science in England. Not only the terminology, inevitably, but equally the content of our early text-books was based directly upon the English models which themselves served as texts in the period from 1650 to 1750 or even later. For this reason details concerning early Oxford, so long the center of mathematics in England, are of particular interest to us.

The illustrations of this work are fine and well chosen; the lists of the early mathematicians and astronomers of Oxford are a welcome addition to the literature of the subject. Unfortunately the text (pp. 1–33) reveals an amazing lack of familiarity with recent publications on the history of mathematics, abounding in errors which could have been avoided by reference to books on the shelves of the Bodleian. However, the descriptive catalogue of early mathematical instruments belonging to the University and Colleges of Oxford, the real *raison d'être* of the work in question, is made with great ability and care. This material constitutes a most worthy addition to our knowledge of mathematical instruments which played so large a part in the discovery and exploration period of American history.

It is not worth while to enumerate the errors on pages 1–33. The statements concerning the sources of Recorde's text-books illustrate the careless editing of the work. Recorde's *Ground of Arts* was not "based on manuscripts entitled *De origine artium* and *Arithmeticae principia*" nor was his *Pathway* (sic) *to Knowledge* "printed from manuscripts entitled *Geometriae semita* and *Theoremata Geometriae*." Neither was the "Whetstone of Witte" "printed . . . from his Secunda pars Arithmeticae." These works were all in English and the material is that current in continental texts of the same period. In passing, it must be said that Recorde

was neither one of the "earliest" nor yet one of the "greatest mathematicians" of England.

In spite of the introduction which should have been prepared by one more familiar with the history of mathematics the work is highly to be commended, as eminently worthy a place in mathematical libraries.

Invitation is extended to subscribe to a guarantee fund to cover the cost of plates for Part III, Astronomy. American libraries are urged to send subscriptions to Part III to R. T. Gunther at Magdalen College, Oxford.

L. C. KARPINSKI

Geschichte der Elementar-Mathematik in systematischer Darstellung. By Johannes Tropfke. Berlin and Leipzig, Vereinigung Wissenschaftlicher Verleger. Bd. I: *Rechnen.* 1921. vi + 177 pp. Bd. II: *Allgemeine Arithmetik.* 1921. 221 pp. Bd. III: *Proportionen. Gleichungen.* 1922. 151 pp. Bd. IV: *Ebene Geometrie.* 1923. 238 pp.

The first edition of Tropfke's *Geschichte der Elementar-Mathematik* was published in 1902–1903, and was reviewed by J. W. A. Young in this BULLETIN.* The four volumes under review form part of a second edition, which is to include seven volumes, the last three being yet in press, or at least not yet available in this country. That the revision has been thoroughgoing is evidenced by the fact that the material corresponding to these four volumes occupied approximately 510 pages in the first edition, and has thus been expanded above 50 per cent; while the references to the literature in these four volumes number 4348, as compared with 1951 in the corresponding parts of the first edition.

And the advance made beyond the first edition is by no means merely quantitative. The author was fortunate in having the active assistance of G. Eneström and H. Wieleitner in the preparation of the new edition, and their names are a sufficient guarantee that no pains have been spared to make the work as complete and authoritative as possible. On nearly every page we find valuable additions to the information given in the first edition; while in several cases the point of view then adopted has been radically changed or even reversed.

To mention but a few of the changes: (1) In discussing the origins of the number system, use has been made of recent researches as to the knowledge of the Babylonians. An interesting detail is the fact (I, p. 15) that as early as 2500 B. C. the tables of Senkereh contain representations of very large numbers according to the sexagesimal system, the largest to date deciphered being $60^8 + 10 \cdot 60^7$ (= 195 955 200 000 000). (2) The history of the development of technical terms is very largely expanded. (3) The independence of the work of the Hindus is in many cases questioned or denied in the second edition where it was accepted in the first; the researches of G. R. Kaye are largely responsible for the author's change of view here. (4) The account of complex numbers (II, 79–90) is entirely rewritten and considerably enlarged. (5) The discussion of the development of the theory of parallels (IV, 53–60) is a considerable improvement over that in the first edition, while it must be confessed that it still leaves something to be de-

* Vol. 12 (1905), pp. 138–140.

sired; thus Saccheri is only referred to in a footnote, and Lambert is not mentioned at all. (6) The work of several individuals has evidently been more thoroughly investigated; thus Leonardo of Pisa's monumental work is more adequately described than in the first edition, while at the same time further information is given as to his Arabian sources. Apparently, however, Abu Kamil has been overlooked among these.* (7) The discussion of indeterminate equations is expanded from 10 to 15 pages (III, 99–114). Here, as well as under the topic "properties of integers" (I, 93–110), we miss any reference to Dickson's *History of the Theory of Numbers*, which would have been of great value to the author, but which presumably was not available during the work of revision.

Two topics which the reviewer would have been glad to see included but which do not seem to be mentioned are the "regula falsi," and the so-called "Russian peasant method" of multiplication. A temporary drawback is the lack of an index, as that is to appear in the last volume. This is of itself a sufficient reason why the publication of the remainder of the work will be awaited with impatience.

A few minor slips or misprints were noted: In referring to the use of the word "cipher" for "zero" (I, 9–10) the author implies that this use is obsolete in English. The date of Leonardo of Pisa's birth is not known to be 1180 (III, 121). The credit for the proof of Steiner's Malfatti problem construction is given to Schroeter, 1874 (IV, 126), whereas it should be given to A. S. Hart, who published a proof in the QUARTERLY JOURNAL OF MATHEMATICS in 1857 (vol. I, pp. 219–221). Misprints occur several times as to dates; thus Albert Girard died in 1632, and the date is so printed in volume I, page 7, but thereafter 1633 is several times found. (II, 76, III, 95, and IV, 123, for example.) Roger Cotes was born in 1682, instead of 1652, as stated on page 159 of volume II. The correct date is given at page 213 of the same volume. There is a misprint on page 134 of volume II, referring to the "fragments of Kahun," which owing to the lack of an index is not easy to correct. The first reference in the "Zeittafel," volume III, page 115, should be I, 128, instead of I, 24. (I, 24 is correct if referring to Leonardo of Pisa's *Scritti*.)

It would be desirable that a reference work of this sort, so near to complete accuracy, should be made even more valuable and kept up to date by the publication at intervals of additions and corrections which might be contributed by anyone interested; possibly the AMERICAN MATHEMATICAL MONTHLY might serve as the medium for such publication.

Tropfke's style is clear and lucid, enlivened by imagery where this can be used without detracting from accuracy. Biographical details are entirely omitted; had they not been, the size of the work must have been expanded to impossible proportions. The topical arrangement brings advantages which fully compensate for the lack of unity which it necessitates. From every point of view the revised edition must be pronounced a decided success, so far as the four volumes permit a judgment of the whole. It can be recommended most warmly to everyone who is interested in the history of elementary mathematics.

R. B. McCLENON

* Cf. L. C. Karpinski, *The Algebra of Abu Kamil*, AMERICAN MATHEMATICAL MONTHLY, vol. 21 (1914), pp. 37–48.

Einführung in die Mechanik Deformierbarer Körper. By Max Planck. Zweite Auflage. Leipzig, Hirzel, 1922. 191 pp.

The first edition of this little book appeared in 1919. This second edition is little more than a corrected first edition. The contents are divided into three parts. Part one (two chapters) deals with strains and stresses in an elastic medium. Part two (four chapters) deals with infinitesimal deformations and in particular with the relations connecting the coefficients of strain and stress. In the last two chapters the preceding theory is applied to the problem of the vibration of solids and fluids. Part three (four chapters) is devoted to rotational and irrotational fluid motion. The last chapter is on viscosity. The equations of hydrodynamics are derived as special cases of the general equations of elasticity. The equation of continuity is given in two forms, one stating that a given small mass of fluid remains invariant, and the other that the difference between the inflow and outflow from a small volume is equal to the increase in the mass contained in the small volume.

The author being a physicist, the treatment is more physical than the usual book on elasticity and hydrodynamics, with the exception of that by Thompson and Tait The style is clear, concise, and appealing. Even one familiar with the subject will find Planck's treatment refreshing and fascinating.

C. L. E. Moore

Lehrbuch der darstellenden Geometrie für technische Hochschulen. By Dr. Emil Müller. Zweiter Band, dritte Auflage. Leipzig and Berlin, B. G. Teubner, 1923. x + 362 pp.

Vorlesungen über darstellende Geometrie. By Dr. Emil Müller. Band I. *Die linearen Abbildungen.* Revised by Dr. Erwin Kruppa. Leipzig and Vienna, Franz Deuticke, 1923. xi + 292 pp.

The first edition of Professor Müller's text on descriptive geometry was reviewed by Professor Snyder [vol. I (1908), 1st part of vol. II (1912), and 2d part of vol. II (1916) in this Bulletin, vol. 16, p. 136; vol. 20, p. 253; and vol. 24, p. 257]. The second editions of the two parts of the second volume, which appeared in 1919, were practically the same as the first edition. In the preface to the third edition of volume II (where the two parts are bound together), the author says that he has made only slight changes, although he has much new material which he might have inserted.

In the preface to the Müller-Kruppa book, Professor Müller says that soon after he took the chair of descriptive geometry (in 1902) at the Technical High School of Vienna, he began to supplement his regular teaching by advanced lectures containing some results of his own researches. These gradually settled down to a four-year cycle, although the content has naturally undergone changes. As he did not have the time to get any of this material into shape for publication, he was happy when his former pupil and present colleague was willing to undertake the task for some of the lectures. Although Professor Kruppa had the original manuscripts, Professor Müller gives him the credit for the final wording, for the drawings, for the revision of some parts of the manuscript, and for certain additions.

This volume consists of three parts: I, Central projection; II, The principles of linear representation; III, Special representations. It also contains a subject index and a name index. In the second part the three principles are called "Zweispuren," "Zweibilder," and "achsonometrische." Ten pages are devoted to Pohlke's theorem. In the third part the four chapters deal with the ordinary linear representations, relief perspective, a representation of plane motion in point space (kinematic representation of Blaschke and Grünwald), and a projective generalization of Lie's line-sphere transformation. Four of the ten chapters of the book are concluded by lists of exercises.

This volume is not intended for beginners: it presupposes a knowledge of both descriptive and projective geometry. But for one who is prepared to read it, it offers an excellent treatment of interesting and valuable material.

E. B. COWLEY

Methodik des mathematischen Unterrichts. 2te durchgesehene und vermehrte Auflage. Zweiter Teil: Didaktik der einzelnen Gebiete. By W. Lietzmann. Leipzig, Quelle und Meyer, 1923. xii + 367 pp.

This work covers the customary mathematical subjects from arithmetic to analytic geometry, the elements of modern geometry, and of calculus, inclusive. In an easy, almost chatty style, the author discusses, at will, points of subject matter, of method and of twentieth century developments in the teaching of the topic in hand. References to twentieth century publications are liberally furnished, to German works as a rule, but also to some in French and a few in English. From arithmetic to analytic geometry, there are given numerous pictures, often from photographs, illustrating models and apparatus. Mathematical moving pictures are discussed (pp. 126–8), and the upshot of the matter is that films have not yet established themselves as valuable adjuncts to mathematical instruction. The illustrations must speak for themselves; no description of the manner of construction or *modus operandi* is given. Nevertheless, for the average American reader these illustrations will perhaps be the most interesting and suggestive portion of the book. They certainly are the most easily accessible.

J. W. A. YOUNG

La Relativité Vraie et la Gravitation Universelle. By Georges Fournier. Paris, Gauthier-Villars, 1923. viii + 130 pp.

This alleged refutation of the work of Einstein must be classed with the work of the circle-squarers. Since those teaching the theory of relativity might use this pamphlet as material for an exercise in locating fallacies, I will not list them here.

It is disappointing to find the imprint of Gauthier-Villars on such a book. This is not the only recently published French work which has suggested the intrusion of a spirit of nationalism into the scientific world.

C. N. REYNOLDS, JR.

Weber-Wellstein, Encyklopädie der Elementarmathematik. Erster Band: *Arithmetik, Algebra und Analysis.* By Heinrich Weber. Vierte Auflage neubearbeitet von Paul Epstein. Leipzig und Berlin, B. G. Teubner, 1922. xvi + 568 pp.

The first edition of this standard and valuable work was reviewed in this BULLETIN (vol. 10, p. 200) by David Eugene Smith. A brief notice of the third edition was published by F. W. Owens (BULLETIN, vol. 17, p. 546). Since the publication of the latter, both the original authors have died. The new (fourth) edition of the first volume appears under the editorship of Paul Epstein. A critical estimate of such a standard work is unnecessary, especially so since no radical changes appear to have taken place. The most noteworthy is the omission of the short chapter on differentiation, which was introduced first in the second edition, and which is now again abandoned partly to save space for material that appears to the present editor more important and partly because a mere brief introduction is not in keeping with plan of the work. An extended treatment was in any case out of the question. Opinions will probably differ as to the validity of the reasons assigned for this omission; but to the present reviewer they appear altogether reasonable. The question as to the meaning to be attached to the term "elementary mathematics" is in any case a difficult one. The meaning accepted by the present author seems to be at least as reasonable as any other. The present edition is in spite of the omission noted somewhat larger than the previous one, owing to the inclusion of some new material and the elaboration of some of the older material. The preface to the present edition indicates these additions, so that they need not be listed here. The historical and bibliographical notes seem to have been distinctly improved. There can be no doubt that the new edition will adequately fill the place in our standard reference literature which the earlier editions filled with such marked success.

J. W. YOUNG

Über die Hypothesen welche der Geometrie zu Grunde liegen. By B. Riemann. Neu herausgegeben und erläutert von H. Weyl. Second edition. Berlin, Julius Springer, 1921. vi + 47 pp.

This reprint of Riemann's famous *Habilitationsschrift* will be welcomed by many. It is one of those few papers which are of permanent and fundamental interest to all who are concerned with the consideration of the general foundation and formulation of fundamental concepts. It is of special value at the present time when the theory of relativity is demanding so much attention. In the last section of his paper Riemann exhibits an almost uncanny insight which Einstein's results finally corroborate. Weyl's notes are very helpful, not only in carrying out in some detail a number of the analytic computations which in Riemann's text are only indicated, but also in giving significant references to later related literature. These notes occupy twenty-four of the forty-seven pages of the pamphlet.

J. W. YOUNG

NOTES

The second number of vol. 24 of the TRANSACTIONS OF THIS SOCIETY (September, 1922) contains: *Generalized limits in general analysis, first paper*, by C. N. Moore; *Anharmonic polynomial generalizations of the numbers of Bernoulli and Euler*, by E. T. Bell; *New properties of all real functions*, by Henry Blumberg; *A fundamental system of invariants of a modular group of transformations*, by J. S. Turner; *The Gaussian law of error for any number of variables*, by J. L. Coolidge; *Certain theorems relating to plane connected point sets*, by Anna M. Mullikin.

Professor Edward Kasner has been elected a member of the Editorial Committee of the TRANSACTIONS OF THIS SOCIETY, as successor to Professor L. P. Eisenhart. Papers in geometry offered to the TRANSACTIONS should be sent to Professor Kasner. Professor Olive C. Hazlett and Dr. Einar Hille have been appointed associate editors.

During the suspension of publication of the BIBLIOTHECA MATHEMATICA, corrections to Cantor's *Vorlesungen über Geschichte der Mathematik* will appear in the JAHRESBERICHT DER DEUTSCHEN MATHEMATIKER-VEREINIGUNG.

The Vienna Academy of Sciences offers a prize for a memoir on *Fictions in mathematics*, at the suggestion of the Society of Friends of the Philosophy of "Als Ob," which has placed the sum of one million marks at the disposal of the Academy for this purpose. Competing memoirs should be written in German, and should be sent to the Academy before December 31, 1925.

Dr. Mallison, Master of Clare College, Cambridge, has offered £500 for a prize, to be called the Mayhew Prize, to be awarded by examination in Part II of the (Cambridge) Mathematical Tripos, to the candidate who in the opinion of the examiners is of greatest merit, preference being given, other things being equal, to candidates whose main subjects are in branches of applied mathematics.

Professors A. Kneser, of the University of Breslau, and E. Study, of the University of Bonn, have been elected corresponding members of the Prussian Academy of Sciences.

Professor E. Strömgren, of the University of Copenhagen, has been called to a professorship of theoretical astronomy at the University of Berlin.

Professor G. Hergoltz, of the University of Leipzig, has been called to a professorship at the University of Munich.

Professor E. Hilb has been promoted to a full professorship at the University of Würzburg.

Dr. F. Böhm has been appointed to an associate professorship at the University of Munich.

Dr. K. Löwner has been admitted as privat docent in mathematics at the University of Berlin.

Professor Niels Bohr, of the University of Copenhagen, will deliver the Silliman lectures at Yale University, and also the Simpson lectures at Amherst College during the present academic year.

Professor J. M. Willard, head of the department of mathematics at Pennsylvania State College, has retired from active service.

Professor F. N. Cole, of Columbia University, ex-secretary of the American Mathematical Society, has been granted leave of absence for the second half of the present academic year.

Dr. Irwin Roman, of Northwestern University, has been appointed associate professor of mathematics at Vanderbilt University.

Mr. A. D. Campbell, of Cornell University, has been appointed assistant professor of mathematics at the University of Arkansas.

Professor O. W. Albert, of Grinnell College, has been appointed head of the department of mathematics at the University of Redlands.

Mr. D. L. Holl, of the University of Chicago, has been appointed assistant professor of mathematics at Ohio Wesleyan University.

Dr. G. M. Robison, of Cornell University, has been appointed assistant professor of mathematics at Trinity College, Durham.

Dr. G. E. Raynor, of Princeton University, has been appointed assistant professor of mathematics at Wesleyan University.

At Princeton University, Dr. C. E. Hille and Dr. C. C. MacDuffee have been promoted to assistant professorships of mathematics.

The two Benjamin Peirce Instructorships in Mathematics at Harvard University (see this Bulletin, vol. 21, page 315) are again open to general competition. Applications for the year 1924–25, accompanied by the necessary papers, should reach Professor Birkhoff not later than February 15, 1924, and all inquiries relating to these appointments should be addressed to him.

Professor J. A. Cragwall, head of the department of mathematics at Wabash College, has been granted a year's leave of absence because of poor health. His duties are being temporarily taken over by Professor G. E. Carscallen.

Professor L. Krüger, formerly director of the Geodetic Institute at Potsdam, died June 1, 1923, at the age of sixty-five years.

Professor Oskar von Peinthner-Lichtenfels, of the Technical School of Graz, died June 9, 1923.

Professor C. N. Little, dean of the college of engineering of the University of Idaho, died September 7, 1923. Professor Little was known for his contributions to the theory of knots.

Professor D. T. Wilson, of the department of astronomy of Case School of Applied Science, died October 12, 1923, in Washington, D. C.

Charles Proteus Steinmetz, professor of electrophysics in Union College and chief consulting engineer for the General Electric Company, died in Schenectady on October 26, 1923, from heart failure, at the age of fifty-eight years. He had been a member of the American Mathematical Society since 1891.

NEW PUBLICATIONS

PART I. PURE MATHEMATICS

ARCHIMEDES. Die Quadratur der Parabel. Ueber das Gleichgewicht ebener Flächen oder über den Schwerpunkt ebener Flächen. (Ostwald's Klassiker der Exakten Wissenschaften.) Leipzig, Akademische Verlagsgesellschaft, 1923. 64 pp.

BESSEL-HAGEN (E.). See KLEIN (F.).

BIANCHI (L.). Lezioni sulla teoria dei numeri algebrici. Pisa, Spoerri, 1923. 6 + 641 pp.

DESCARTES (R.). Geometrie. Deutsch herausgegeben von L. Schlesinger. 2te, durchgesehene Auflage. Leipzig, Mayer und Müller, 1923. 11 + 121 pp.

FRANK (E.). Plato und die sogenannten Pythogoreer. Ein Kapitel aus der Geschichte des griechischen Geistes. Halle, Verlag von Max Niemeyer, 1923. 10+ 400 pp.

FRICKE (R.). See KLEIN (F.).

GILLMER (M.). Trigonometrie und Stereometrie. Leipzig, Hirzel, 1922. 6 + 226 pp.

IMBODEN (A.). Resumen de geometria proyectiva elemental. Paraná, Prometeo, 1923. 22 pp.

KLEIN (F.). Gesammelte mathematische Abhandlungen. Band 3: Elliptische Funktionen, insbesondere Modulfunktionen, Hyperelliptische und abelsche Funktionen, Riemannsche Funktionentheorie und automorphe Funktionen. Anhang: Verschiedene Verzeichnisse. Herausgegeben von R. Fricke, H. Vermeil, und E. Bessel-Hagen. Berlin, Springer, 1923. 10 + 774 + 36 pp.

LEVY (H.). See RUNGE (C.).

LOVE (C. E.). Analytic geometry. New York, Macmillan, 1923. 14 + 306 pp.

VON MANGOLDT (H.). Einführung in die höhere Mathematik. 4te Auflage. Band I. Leipzig, Hirzel, 1923.

MAYER (O.). Contributions à la théorie des quartiques bicirculaires. (Thèse, Jassy.) Jassy, Imprimerie H. Goldner, 1923. 99 pp.

NEYMEYER (L.). Zur Theorie der Krümmungsmasse mehrdimensionaler Mannigfaltigkeiten. (Diss.) Freiburg i. Br., 1922.

ONNEN (H.). Kreisevolventen und algebraische Funktionen. Leipzig, Teubner, 1923. 43 pp.

PASCALI (J.). Calcolo de diferencias finitas. Buenos Aires, Revista del Centro de Estudiantes de Ingenieria, 1922. 16 pp.

PICARD (E.). La vie et l'œuvre de Pierre Duhem. Paris, Gauthier-Villars, 1922. 4to. 58 pp.

RUNGE (C.). Vector analysis. Translated by H. Levy. London, Methuen, 1923. 8 + 226 pp.

SCHLESINGER (L.). See DESCARTES (R.).

SCHRUTKA (L.). Zahlenrechnung. Leipzig, Teubner, 1923. 10 + 146 pp.

VERMEIL (H.). See KLEIN (F.).

PART II. APPLIED MATHEMATICS

ANDRADE (E.). The structure of the atom. London, Bell, 1923. 15 + 314 pp.

BIDWELL (C. C.). Outlines of physics. Ithaca, Comstock Publishing Company, 1922. 104 pp.

BOND (P. S.). Map reading and military sketching. New York, Army and Navy Journal, 1922. 104 pp.

BOUASSE (H.). Dynamique générale. Paris, Delagrave, 1923. 324 pp.

BRILLOUIN (H.). Théorie électrique moderne de l'état solide. Paris, Société française des Electriciens, 1923. 8vo. 24 pp.

CODD (L. W.). See NERNST (W.).

DINGLER (H.). Das Problem des absoluten Raumes. Leipzig, Hirzel, 1923. 50 pp.

ENCYCLOPÄDIE der mathematischen Wissenschaften. Band V 2, Heft 5: R. Seeliger, Elektronentheorie der Metalle. Leipzig, Teubner, 1922.

HOPKINS (M.). Chance and error. The theory of evolution. New York, Dutton, and London, Kegan Paul, Trench, Trübner, 1923. 8 + 223 pp.

KADERAVEK (F.). Perspektiva. Prague, J. Stene, 1922. 109 pp. + 31 tables.

KIRBY (R. S.). Exercises in the elements of descriptive geometry. New York, Wiley, 1922. 10 + 49 pp.

KOBOLD (H.). See MÖBIUS (A. F.).

LARNER (E. T.). Alternating currents, their theory and transmission. London, Lockwood, 1922. 8 + 198 pp.

VON LAUE (M.). Ueber die Auffindung der Röntgenstrahlinterferenzen. Nobelvortrag. Karlsruhe, C. F. Müller, 1920. 16 pp.

LEWIS (G. M.) and RANDALL (M.). Thermodynamics and free energy of chemical substances. London, McGraw-Hill, 1923. 23 + 653 pp.

MARAIS (H.). Introduction géométrique à l'étude de la relativité. Paris, Gauthier-Villars, 1923. 8vo. 192 pp.

MIE (G.). Die Einstein'sche Gravitationstheorie. 2te Auflage. Leipzig, Hirzel, 1923. 4 + 69 pp.

MÖBIUS (A. F.). Astronomie. Grösse, Bewegung und Entfernung der Himmels-Körper. 13te Auflage, bearbeitet von H. Kobold. Teil 2. Berlin, Vereinigung wissenschaftlicher Verleger, 1923. 128 pp.

MOREUX (T.). La science mystérieuse des Pharaons. Paris, Doin, 1923. 16mo. 250 pp.

NERNST (W.). Theoretical chemistry from the standpoint of Avogadro's rule and thermodynamics. 5th edition, revised by L. W. Codd. London, Macmillan, 1923. 20 + 922 pp.

OLLIVIER (H.). Cours de physique générale. Tome III. 2e édition, entièrement refondue. Paris, Hermann, 1923. 712 pp

RANDALL (M.). See LEWIS (G. M.).

SEELIGER (R.). See ENCYKLOPÄDIE.

THIRTY-SECOND ANNUAL LIST OF PAPERS

READ BEFORE THE AMERICAN MATHEMATICAL SOCIETY AND SUBSEQUENTLY PUBLISHED, INCLUDING REFERENCES TO THE PLACES OF PUBLICATION

ALEXANDER, J. W. A proof and extension of the Jordan-Brouwer separation theorem. Read April 29, 1916. *Transactions of this Society*, vol. 23, No. 4, pp. 333–349; June, 1922.

—— Invariant points of a surface transformation of given class. Read Dec. 28, 1922. *Transactions of this Society*, vol. 25, No. 2, pp. 173–184; April, 1923.

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THE SAN FRANCISCO SECTION, AT THE UNIVERSITY OF WASHINGTON, IN SEATTLE, December 22, 1923. See also this Bulletin, vol. 29, No. 9 (Nov., 1923), p. 425.

Abstracts must be in the hands of the acting Secretary of the Section, D. N. Lehmer, Berkeley, Cal., not later than November 29.

THE ANNUAL MEETING OF THE SOCIETY, IN NEW YORK CITY, December 27–28, 1923. See also this Bulletin, vol. 29, No. 4 (April, 1923), p. 188.

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THE TWENTIETH WESTERN MEETING OF THE SOCIETY (Fifty-second Meeting of the Chicago Section), in Cincinnati, December 28–29, 1923. See also this Bulletin, vol. 29, No. 5 (May, 1923), p. 197; and No. 9 (Nov., 1923), p. 425.

Abstracts must be in the hands of the Secretary of the Chicago Section, Arnold Dresden, 2114 Vilas St., Madison, Wis., not later than November 29.

NEW YORK CITY, February 24, 1924.

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THE SAN FRANCISCO SECTION, AT STANFORD UNIVERSITY, April 5, 1924.

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